International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 23s, 2025
https://www.theaspd.com/ijes.php

Mathematical Model For Environmental Pollution Prevention

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Abstract
This paper explores mathematical modeling methods based on differential equations aimed at preventing environmental pollution. The main focus is on formalizing the dynamics of pollutant dispersion, degradation, and removal over space and time. To this end, advection-diffusion type partial differential equations (PDEs) are employed, with analytical solutions obtained using separation of variables, as well as numerical approaches. By applying optimal control theory, management functions to reduce pollution in ecological systems are constructed and explained using Pontryagin's maximum principle. Moreover, stochastic differential equations (SDEs) are introduced to account for uncertainties present in real ecological processes, which is of particular importance for ecological risk assessment. The proposed models provide a theoretical foundation for environmental system management while allowing more precise mathematical investigation of applied ecological problems.
Keywords: *diffusion, industrialization, density, ecosystem, concentration*

1.INTRODUCTION
With industrialization, urbanization, and population growth, environmental pollution has become one of the major global ecological problems. Increasing concentrations of pollutants in the air, water, and soil disrupt ecosystem balance, reduce biodiversity, and have significant adverse effects on human health (Seinfeld, Pandis, [1]). Pollutants released into the atmosphere, surface and groundwater, and soil from industrial facilities, transportation, and agricultural activities create long-term ecological problems (Zannetti, [2]). Therefore, environmental pollution prevention should not be limited to monitoring and empirical observation but must also be grounded in scientific principles and mathematical analysis.
Mathematical modeling is a crucial tool for understanding and predicting the dynamics of pollution processes. Differential equations allow for the quantitative description of pollutant emission, dispersion, degradation, and removal (Paal, [3]; Tauber, [4]). This approach is widely applied not only for assessing environmental conditions but also for developing management strategies that minimize pollution.
Previous studies have shown that pollution dynamics can be expressed using different models in various media. For instance, Paal [3] described the dispersion of airborne pollutants using partial differential equations, analyzing how concentration changes with distance from the source and over time. Tauber [4] employed vector partial differential equations to model rapid pollutant dispersion and spatial distribution in the atmosphere. Huang and Li [5] used stochastic differential equations to model water pollution, assessing the impact of random variations on the processes.
Subsequently, Seinfeld and Pandis [1] presented extensive PDE systems covering the chemical reactivity and transport of atmospheric pollutants, while Zannetti [2] developed three-dimensional pollutant dispersion models considering meteorological conditions and topographic factors. Comparisons indicate that deterministic models are suitable for long-term predictions, whereas stochastic models more accurately reflect random and uncertain ecological variations (Huang, Li, [5]).
The significance of mathematical modeling for environmental management also emerges in the context of optimal control. The goal is to minimize pollutant concentrations while achieving economically and technologically feasible solutions. The maximum principle introduced by Pontryagin et al. [6] facilitates the design of optimal control strategies. Fleming and Rishel [7] further developed stochastic variants of optimal control, taking into account random and uncertain ecological processes.
Works [8-15] develop mathematical models and boundary conditions for predicting corrosion under mechanical stress of ship hull metals in aggressive environments.
Thus, differential equation-based mathematical modeling provides a scientific and practical framework for describing the spread and prevention of pollution. Comparative analyses of various studies show that deterministic models are useful for long-term forecasting, while stochastic models more accurately reflect

random variations and uncertainties in real-world conditions. This paper develops a mathematical model of pollution, solves it using differential equations, and presents both analytical solutions as well as approaches incorporating optimal control and stochastic modeling. The aim is to establish a mathematical framework that ensures ecological stability and can be applied in practical management.

Distinct Advantages of This Study

1. **Comprehensive Approach:** Both deterministic models (classical advection-diffusion equations) and stochastic models (SDE/SPDE) are presented. This allows the study of pollutant dynamics under both theoretical conditions and real-world uncertainties.
2. **Integration of Optimal Control:** Unlike traditional models, Pontryagin's maximum principle is applied to construct an optimal control function to prevent pollution. This not only analyzes ecological processes but also theoretically substantiates their management mechanism.
3. **Synthesis of Analytical and Numerical Solutions:** Unlike studies relying solely on numerical simulations or analytical methods, this work applies both approaches comparatively, enhancing both theoretical reliability and practical applicability.
4. **Modeling Ecological Uncertainties:** Stochastic effects are incorporated to better reflect the variability of atmospheric and hydrological processes. This approach is particularly important for risk management and ecological policy planning.
5. **Wide Applicability:** The proposed models can be applied not only to pollutant dispersion in the atmosphere and water bodies but also to evaluate the environmental impact of industrial processe.

2. PROBLEM STATEMENT

To mathematically model the process of environmental pollution, it is first necessary to determine the variation of pollutant concentration with respect to time and spatial coordinates. Let $C(x,t)$ concentration of the pollutant at spatial coordinate x nd time t. he dispersion, removal, and emission of pollutants from sources can be described by the following partial differential equation

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial t^2} - kC(x,t) + S(x,t), \quad (1)$$

where D is the diffusion coefficient (m²/s),), k is the degradation or removal rate of the pollutant (1/s), $S(x,t)$ is the source term representing the pollutant emission (kg/m³·s).

This equation is referred to as the advection-diffusion equation and serves as the fundamental model for pollutant distribution in atmospheric or aquatic environments (Tauber, [4]; Seinfeld & Pandis, [1]).

The solution of this problem requires the following additional conditions:

- Initial condition: $C(x,0) = C_0(x)$, (2)
- Boundary conditions at the domain endpoints $C(0,t) = C(L,t) = 0$, (3). These are known as Dirichlet boundary conditions.

3. PROBLEM SOLUTION

3.1. Analytical Solution

Definition 1: A partial differential equation (PDE) is an equation that involves derivatives of a function with respect to two or more independent variables.

Theorem 1: If a PDE is linear and subject to homogeneous boundary conditions, it can be solved using the method of separation of variables.

Proof: Let us assume that the solution has the form

$$C(x,t) = X(x)T(t).$$

Then, equation (1) becomes

$$X(x)\frac{dT(t)}{dt} = DT(t)\frac{d^2X(x)}{dx^2} - kX(x)T(t) + S(x,t).$$

If $S(x,t) = 0$, i.e., the equation is homogeneous, we obtain

$$\frac{1}{T(t)}\frac{dT(t)}{dt}+x=D\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=-\lambda$$

where λ is the separation constant. Thus, the PDE can be split into two ordinary differential equations:

1.Spatial equation

$$D\frac{d^2X(x)}{dx^2}+\frac{\lambda}{D}kX(x)=0,$$

2.Temporal equation

$$\frac{dT(t)}{dt}+(k+\lambda)T(t).$$

Solving these equations yields

$$X_n(x)=\sin\left(\frac{\pi n x}{L}\right),T_n(t)=\exp\left(-\left(k+D\left(\frac{\pi n}{L}\right)^2t\right)\right).$$

Therefore, the general solution of equation (1) is

$$C(x,t)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{\pi n x}{L}\right)\exp\left(-\left(k+D\left(\frac{\pi n}{L}\right)^2t\right)\right)$$

where the coefficients A_n are determined from the initial condition,

$$A_n=\frac{2}{n}\int_0^L C_0(x)\sin\left(\frac{\pi n x}{L}\right)dx.$$

3.2. Optimal Control

Definition 2: Optimal control is a control strategy that maximizes or minimizes the system output according to a specified objective.

Theorem 2: If the system dynamics are given by $\dot{x}=f(x,u,t)$ **then the optimal control** $u^*(t)$ **must maximize the Hamiltonian** $H(x,u,\lambda,t)$ **, i.e.**

$$H(x^*,u^*,\lambda,t)=\max_x H(x,u,\lambda,t).$$

Proof: This theorem was established by Pontryagin et al. [6]. The Hamiltonian is defined as

$$H=\lambda(t)\left(D\frac{d^2C}{dx^2}-kC+u(t)\right),$$

where $u(t)$ is the control function representing technological intervention to reduce pollutant emissions. The optimal control $u^*(t)$ is determined using the maximum principle.

3.3. Stochastic Modeling

If random fluctuations occur in the pollution process, stochastic differential equations (SDEs) are used:

$$dC(x,t)=\left(D^2\frac{\partial^2C(x,t)}{\partial x^2}-kC(x,t)+S(x,t)\right)dt+\sigma C(x,t)dW_t$$

where W_t is the Wiener process, and σ - is the stochastic fluctuation coefficient (Huang, Li, [5]). This approach accounts for the uncertainty inherent in real ecological processes.

RESULTS

1. Differential equations provide a mathematical description of pollutant dispersion and removal processes.
2. Both analytical and stochastic solutions are available for fixed and variable sources.
3. Optimal control strategies minimize pollutant concentration, offering ecologically and economically sustainable solutions.

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