International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 7, 2025
https://www.theaspd.com/ijes.php

Existence And Uniqueness Results For Delay Integral Equations In B-Metric Spaces

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Abstract
This paper establishes a novel framework for analyzing nonlinear Volterra integral equations with hereditary effects within complete b-metric spaces. We investigate the existence and uniqueness of solutions to equations of the form: $x(t) = g(t) + \lambda \int_a^b K(t,s)\Phi\left(s, x(s), \int_{s-\tau}^{s} x(u)du\right)ds$ *under Lipschitz continuity and local boundedness conditions. Our approach introduces explicit a priori bounds and verifies operator continuity, constructing invariant sets where Agrawal's fixed-point theorem applies with computable constants satisfying* $k + 2cs < 1$. *The theoretical advancements address significant limitations in global boundedness assumptions while maintaining verifiable convergence criteria. Practical applications to population dynamics and RLC circuits demonstrate the framework's efficacy, with explicit radius calculations ensuring consistency with hereditary system constraints.*
Keywords: *Fixed point theorem, b-metric space, Volterra integral equation, hereditary systems, invariant set*

1. INTRODUCTION

Fixed point theory in generalized metric spaces provides fundamental tools for the analysis of functional equations exhibiting hereditary effects, where past states influence present behavior. In particular, the relaxation of the triangle inequality in b-metric spaces, with $s \geq 1$ as introduced by Bakhtin [2], allows the treatment of systems in which classical metric structures fail to capture memory-dependent interactions, as discussed in several studies [13, 14]. This generalization has proven especially valuable for integral equations modeling biological, physical, and engineering systems with time-lagged responses [15], thereby extending the applicability of fixed point techniques to complex real-world phenomena.

Recent developments in fixed point theory have further expanded classical results to more general frameworks. Agrawal and collaborators [1] proposed an innovative contractive condition that not only generalizes Banach's classical principle [3] but also accommodates geometric complexities inherent in systems with hereditary properties [10, 11]. Building upon earlier work by Czerwik [6, 7] and others, these developments have created a rich theoretical foundation for analyzing nonlinear systems. Specifically, the contractive inequality

$$d(Tx,Ty) \leq k\max\{d(x,Tx), d(y,Ty), d(x,y)\} + c[d(x,Ty) + d(y,Tx)],$$

with the constraint $k + 2cs < 1$, enables a more flexible approach to contractions, which is particularly relevant when analyzing systems where memory effects introduce asymmetries in operator behavior [11] .

Despite these theoretical advances, significant challenges remain in applying fixed point results to integral equations involving delay terms. Often, global boundedness assumptions on nonlinear

kernels (Φ, f) are imposed, which may be unrealistic for unbounded domains [12]. In addition, the continuity of associated operators is frequently assumed without rigorous verification [4], potentially undermining the applicability of standard fixed point arguments. This issue has been noted in various contexts, including the work of Bota et al. [4] on Ekeland's variational principle. Moreover, the generalized contractive condition is sometimes underutilized by setting $c = 0$, thereby neglecting scenarios in which a positive c could relax constraints and capture asymmetric memory
effects more effectively [1], a limitation also observed in the multivalued contractions studied by Boriceanu [5] and in the study of almost contractions sequences by Pacurar [9].

This study addresses these challenges by deriving explicit a priori bounds for invariant radii R_0, replacing global boundedness assumptions with locally verifiable conditions. Furthermore, operator continuity is rigorously established through the application of composition principles and the dominated convergence theorem, ensuring that the integral operators under consideration are well-defined. This approach extends beyond conventional methods and incorporates insights from recent advances in almost contraction theory [9] and Ekeland's variational principle in b-metric spaces [4]. In addition, the contractivity conditions are optimized by identifying situations in which the choice of $c > 0$ improves the contraction framework, thereby accommodating systems with pronounced asymmetric memory effects.

The theoretical framework developed herein is illustrated through applications to delayed population dynamics [13] and RLC circuits with memory-dependent charge accumulation 14. By explicitly computing invariant radii R_0 and contractive constants $k = A^p$, the results are rendered computationally verifiable, thus providing a significant advancement in the practical analysis of hereditary systems with time-lagged interactions [15]. The comprehensive nature of this analysis draws upon foundational works in well-known fixed point theorems [8] and nonlinear set-valued contractions [7].

2. Preliminaries

This section presents the foundational concepts necessary for our main results. We begin by recalling the structure of b-metric spaces and their key properties, then state the fixed-point theorems that will be employed in our analysis.

Throughout this paper, we consider the integral equation

$$x(t) = g(t) + \lambda \int_a^b K(t,s)\Phi\left(s, x(s), \int_{s-\tau}^{s} x(u)du\right)ds$$

under the following assumptions:

(1) $g: [t_0 - \tau, t_0] \to \mathbb{R}$ is continuous and bounded with $|g(t)| \leq M_g$.

(2) $\Phi(s, x, \xi)$ is continuous and satisfies the Lipschitz condition:

$$|\Phi(s, x_1, \xi_1) - \Phi(s, x_2, \xi_2)| \leq L_\Phi(|x_1 - x_2| + |\xi_1 - \xi_2|)$$

and is locally bounded: $|\Phi(s, x, \xi)| \leq M_\Phi(R)$ for $\|x\|_\infty \leq R$. The local boundedness assumption (as opposed to global boundedness) is more realistic in applications, as many functions are bounded only on finite intervals.

(3) $K(t,s)$ is continuous and bounded: $|K(t,s)| \leq M_K$.

(4) There exists $R_0 > 0$ such that

$$M_g + |\lambda| M_K M_\Phi(R_0)(b - a) \leq R_0$$

(5) Define

$$A = |\lambda|(b - a)(1 + \tau)M_K L_\Phi$$

(6) Assume $A < 1$ (necessary for existence of $p > 1$ with $A^p < 1$). The condition $A < 1$ ensures that there exists $p > 1$ such that $A^p < 1$, which is necessary for the contraction condition $k + 2cs < 1$ to hold with $k = A^p$ and $c = 0$.

(7) $a = t_0$, ensuring the domain of the delayed term $[s - \tau, s] \subseteq [t_0 - \tau, b]$ for all $s \in [a, b]$.

Remark 2.1. The choice $a = t_0$ for the lower limit of integration is standard in delay Volterra equations. It is not merely a simplifying assumption but a necessary condition for the consistency of the problem. It ensures that for any $s \in [a, b] = [t_0, b]$, the interval of the delayed term $[s - \tau, s]$ is entirely contained within the solution domain $[t_0-\tau,b]$, thus guaranteeing that the term $\int_{s-\tau}^{s} x(u)du$ is well-defined.

Definition 2.2 (b-Metric Space [2]). Let X be a nonempty set and let $s \geq 1$ be a real number. A function

$$d: X \times X \longrightarrow \mathbb{R}^+$$

is called a b-metric if, for all $x, y, z \in X$, the following hold:

(1) (Positive definiteness) $d(x,y)=0$ if and only if $x=y$.

(2) (Symmetry) $d(x,y)=d(y,x)$.

(3) (Relaxed triangle inequality) $d(x,z) \leq s[d(x,y)+d(y,z)]$.

The triple (X,d,s) is called a b-metric space. When $s=1$, this reduces to an ordinary metric space.

Definition 2.3 (Convergence and Completeness in a b-Metric Space [10]). Let (X,d,s) be a b -metric space.

(1) A sequence $\{x_n\} \subset X$ converges to $x \in X$ if

$$\lim_{n\to\infty} d(x_n,x)=0$$

(2) The sequence $\{x_n\}$ is Cauchy if

$$\lim_{m,n\to\infty} d(x_m,x_n)=0$$

(3) The space (X,d,s) is complete if every Cauchy sequence converges to some point in X.

Theorem 2.4 (Fixed Point Theorem in a b-Metric Space 1]). Let (X,d,s) be a complete b-metric space, and let $T: X \to X$ be a mapping for which there exist constants $k, c \geq 0$ satisfying

$$d(Tx,Ty) \leq k\max\{d(x,Tx), d(y,Ty), d(x,y)\} + c[d(x,Ty)+d(y,Tx)], \forall x,y \in X,$$

and assume

$$k+2cs<1.$$

Then T has a unique fixed point in X, and for any initial $x_0 \in X$, the Picard iteration $x_{n+1}=T(x_n)$ converges to this unique fixed point.

3. MAIN RESULTS

This section presents the core theoretical findings of the paper. It formulates the integral equation under study and proves the main theorem which guarantees the existence of a unique solution within an explicitly defined invariant set under computable contractivity conditions.

Lemma 3.1. Let $X=C([t_0-\tau,b],\mathbb{R})$ with the b-metric

$$d(x,y)=\left(\sup_{t\in[t_0-\tau,b]} |x(t)-y(t)|\right)^p, p>1.$$

Then (X,d,s) with $s=2^{p-1}$ is complete.

Proof. First, consider positive definiteness and symmetry. For any $x,y \in X, d(x,y)=0$ if and only if $\sup_{t\in[t_0-\tau,b]} |x(t)-y(t)|=0$, which implies $x(t)=y(t)$ for all $t \in [t_0-\tau,b]$. Moreover, it is evident that $d(x,y)=d(y,x)$, satisfying symmetry.

Next, consider the relaxed triangle inequality. Take any $x,y,z \in X$ and define

$$A=\sup_{t\in[t_0-\tau,b]} |x(t)-z(t)|, B=\sup_{t\in[t_0-\tau,b]} |x(t)-y(t)|, C=\sup_{t\in[t_0-\tau,b]} |y(t)-z(t)|.$$

By the classical triangle inequality of the supremum norm, $A \leq B+C$. Since $p>1$, the function $f(u)=u^p$ is convex on $[0,\infty)$, and applying Jensen's inequality yields

$$(B+C)^p \leq 2^{p-1}(B^p+C^p)$$

Therefore, $A^p \leq (B+C)^p \leq 2^{p-1}(B^p+C^p)=s(d(x,y)+d(y,z))$, which establishes the relaxed triangle inequality with $s=2^{p-1}$.

Finally, completeness is verified as follows. Let $\{x_n\}$ be a Cauchy sequence in (X,d). Then for every $\epsilon>0$, there exists N such that for all $m,n \geq N$,

$$d(x_m,x_n)=\left(\sup_{t\in[t_0-\tau,b]} |x_m(t)-x_n(t)|\right)^p<\epsilon$$

which implies $\sup_{t \in [t_0-\tau,b]} |x_m(t) - x_n(t)| < \epsilon^{1/p}$. Hence, $\{x_n\}$ is uniformly Cauchy in the supnorm. Since $C([t_0-\tau,b])$ is complete under the sup-norm, there exists $x \in X$ such that $x_n \to x$ uniformly. To show convergence in (X,d), observe that for any $\epsilon > 0$, one can choose N such that for $n \geq N$,

$$\sup_{t \in [t_0-\tau,b]} |x_n(t) - x(t)| < \epsilon^{1/p}$$

which gives $d(x_n,x) = \left(\sup_{t \in [t_0-\tau,b]} |x_n(t) - x(t)|\right)^p < \epsilon$. Therefore, $x_n \to x$ in (X,d), and the space is complete.

Lemma 3.2 (Continuity of T). The operator $T: X \to X$ defined by

$$T(x)(t) = g(t) + \lambda \int_a^b K(t,s)\Phi\left(s, x(s), \int_{s-\tau}^s x(u)du\right)ds$$

is continuous.

Proof. To establish continuity of the operator T , we employ the dominated convergence theorem. Let $\{x_n\}$ be a sequence in X converging uniformly to x. We need to show that $T(x_n) \to T(x)$ uniformly.

For each fixed $t \in [t_0-\tau,b]$, consider the difference:

$$|T(x_n)(t) - T(x)(t)| \leq |\lambda| \int_a^b |K(t,s)| \left|\Phi\left(s, x_n(s), \int_{s-\tau}^s x_n(u)du\right) - \Phi\left(s, x(s), \int_{s-\tau}^s x(u)du\right)\right| ds$$

By the Lipschitz condition on Φ and boundedness of K, we have:

$$|T(x_n)(t) - T(x)(t)| \leq |\lambda| M_K L_\Phi \int_a^b \left(|x_n(s) - x(s)| + \left|\int_{s-\tau}^s (x_n(u) - x(u))du\right|\right) ds$$

Since $x_n \to x$ uniformly, for any $\epsilon > 0$, there exists N such that for all $n > N$ and all $u \in [t_0-\tau,b]$, $|x_n(u) - x(u)| < \epsilon$. Thus:

$$|T(x_n)(t) - T(x)(t)| \leq |\lambda| M_K L_\Phi \int_a^b (\epsilon + \tau\epsilon) ds = |\lambda| M_K L_\Phi (b-a)(1+\tau)\epsilon$$

This establishes uniform convergence of $T(x_n)$ to $T(x)$, proving the continuity of operator T .

Remark 3.3. The continuity of the operator T is established here using a direct $\epsilon - \delta$ argument based on the uniform convergence of x_n to x. While the Dominated Convergence Theorem is a common and powerful tool for proving continuity of integral operators, the direct method is more straightforward and is sufficient in this context, given the finite interval $[a,b]$ and the continuity (and hence boundedness) of the kernel $K(t,s)$ on its domain.

Theorem 3.4. Under the stated assumptions, let $T: X \to X$ be the operator defined by

$$T(x)(t) = g(t) + \lambda \int_a^b K(t,s)\Phi\left(s, x(s), \int_{s-\tau}^s x(u)du\right)ds$$

Then, there exists $p > 1$ such that $A^p < 1$. For such a p, the operator T satisfies the contractive condition of Theorem 2.3 with constants $k = A^p$ and $c = 0$, and hence admits a unique fixed point. Moreover, the sequence defined by Picard iteration $x_{n+1} = T(x_n)$ converges to this unique fixed point for any initial guess $x_0 \in X$.

Proof. We first show that T maps the closed ball $B_{R_0} = \{x \in X: \|x\|_\infty \leq R_0\}$ into itself. For any $x \in B_{R_0}$, we have:

$$|T(x)(t)| \leq |g(t)| + |\lambda| \int_a^b |K(t,s)| \left|\Phi\left(s, x(s), \int_{s-\tau}^s x(u)du\right)\right| ds$$

Using the boundedness conditions (assumptions 1,3 , and 4):

$$|T(x)(t)| \leq M_g + |\lambda| M_K M_\Phi(R_0)(b-a) \leq R_0$$

Thus, $\|T(x)\|_\infty \leq R_0$, so $T(x) \in B_{R_0}$.

Since B_{R_0} is closed in the complete space X (by Lemma 3.1), it is itself a complete b-metric space.

Now, for any $x, y \in B_{R_0}$, we estimate:

$$|T(x)(t)-T(y)(t)| \leq |\lambda| \int_a^b |K(t,s)| \left| \Phi\left(s, x(s), \int_{s-\tau}^{s} x(u)du\right) - \Phi\left(s, y(s), \int_{s-\tau}^{s} y(u)du\right) \right| ds$$

Using the Lipschitz condition (assumption 2) and boundedness of :

$$|T(x)(t)-T(y)(t)| \leq |\lambda| M_K L_\Phi \int_a^b (|x(s)-y(s)| + \tau\|x-y\|_\infty) ds$$

Since $\int_a^b ds = b-a$ and $\|x-y\|_\infty$ is constant with respect to s, we obtain:

$$\|T(x)-T(y)\|_\infty \leq |\lambda| M_K L_\Phi (b-a)(1+\tau)\|x-y\|_\infty = A\|x-y\|_\infty$$

Now consider the b-metric $d(x,y) = (\|x-y\|_\infty)^p$. Then:

$$d(T(x),T(y)) \leq A^p d(x,y)$$

Since $A<1$ (assumption 6), we can choose $p>1$ such that $k=A^p<1$. Then:

$$d(T(x),T(y)) \leq kd(x,y) \leq k\max\{d(x,T(x)), d(y,T(y)), d(x,y)\}$$

With $c=0$, we have $k+2cs=k<1$. By Agrawal's theorem (Theorem 2.3), T has a unique fixed point in B_{R_0}, and the Picard iteration converges to this fixed point.

4. Examples and Applications

This section presents two examples illustrating the application of Theorem 3.4, under explicit parameter settings. Both examples are modeled using nonlinear integral equations with memory effects.

4.1. Example: Delayed Logistic Model with Explicit R_0. Consider the population dynamics model with delayed regulation, where the population at time t is given by

$$P(t) = P_0 + \lambda \int_0^T e^{-(t+s)} \left(P(s) + \alpha \int_{s-\tau}^{s} P(u)du\right) ds$$

with parameters $\lambda=0.1, \tau=0.3, T=1, \alpha=0.8, P_0=0.5$, and $M_K=1$, since $\left|e^{-(t+s)}\right| \leq 1$. The nonlinear term and the kernel are defined as $\Phi(s,P,\eta)=P+\alpha\eta$ and $K(t,s)=e^{-(t+s)}$, respectively. For a bounded function P such that $\|P\|_\infty \leq R$, it follows that

$$M_\Phi(R) = \sup|P+\alpha\eta| \leq R + 0.8\cdot(0.3R) = 1.24R$$

The condition for the invariant radius R_0 becomes

$$0.5 + 0.1\cdot 1\cdot(1.24R_0)\cdot 1 \leq R_0$$

which gives

$$0.5 + 0.124R_0 \leq R_0$$

Thus, choosing $R_0=0.7$ satisfies the inequality because $0.5+0.124\times 0.7 \approx 0.587 < 0.7$. The Lipschitz constant of Φ is $L_\Phi = 1+\alpha = 1.8$. Consequently, the contractivity constant is computed as

$$A = 0.1\cdot 1\cdot 1\cdot 1.8\cdot(1+0.3) = 0.234$$

Choosing $p=1.5$ yields $k=A^p \approx 0.113$ and $s=2^{0.5}\approx 1.414$, and therefore,

$$k+2cs = 0.113+0 = 0.113 < 1$$

It follows that all conditions of Theorem 3.4 are satisfied, ensuring the existence of a unique solution in the ball $B_{0.7}$.

4.2. Example: Nonlinear Circuit with Memory and Quadratic Feedback. Consider an electrical system with nonlinear feedback and memory effects, where the voltage $x(t)$ is governed by:

$$x(t) = V_0 + \lambda \int_0^T e^{-(t+s)} \left[x(s) + \delta x(s)^2 + \gamma \int_{s-\tau}^{s} x(u)du\right] ds$$

with parameters $\lambda=0.1, \tau=0.3, T=1, \delta=0.2, \gamma=0.5$, and $V_0=0.5$. Here, the nonlinear function is $\Phi(s,x,\eta) = x+\delta x^2+\gamma\eta$, and the kernel is $K(t,s)=e^{-(t+s)}$, with $|K(t,s)|\leq 1$ so $M_K=1$. The input is constant with $M_g=0.5$. For $\|x\|_\infty \leq R$, we have $|\eta| \leq \tau R = 0.3R$, thus:

$$|\Phi(s,x,\eta)| \leq R + 0.2R^2 + 0.5\cdot(0.3R) = 0.2R^2 + 1.15R$$

Thus, $M_\Phi(R) = 0.2R^2+1.15R$. The invariant radius R_0 must satisfy:

$$0.5 + 0.1\cdot 1\cdot(0.2R_0^2 + 1.15R_0)\cdot 1 \leq R_0$$

which simplifies to:

$$0.02R_0^2 - 0.885R_0 + 0.5 \leq 0$$

Solving this quadratic inequality yields $R_0 \in [0.636, 39.3]$. Choose $R_0 = 0.64$. The Lipschitz constant is computed locally for $|x| \leq R_0$:

$$|\Phi(s, x_1, \eta_1) - \Phi(s, x_2, \eta_2)| \leq \left|(x_1 + 0.2x_1^2) - (x_2 + 0.2x_2^2)\right| + |\gamma||\eta_1 - \eta_2|$$
$$\leq (1 + 0.4R_0)|x_1 - x_2| + \gamma|\eta_1 - \eta_2|$$

Noting that $|\eta_1 - \eta_2| \leq \tau \sup_u |x_1(u) - x_2(u)| = \tau \|x_1 - x_2\|_\infty$, we get:

$$L_\Phi = (1 + 0.4R_0) + \gamma\tau = (1 + 0.4 \times 0.64) + (0.5 \times 0.3) = 1.256 + 0.15 = 1.406$$

The contractivity constant is:

$$A = |\lambda| M_K L_\Phi (b - a)(1 + \tau) = 0.1 \cdot 1 \cdot 1.406 \cdot 1 \cdot (1 + 0.3) = 0.18278$$

Choose $p = 1.1$, so $k = A^p \approx 0.18278^{1.1} \approx 0.168$ and $s = 2^{p-1} = 2^{0.1} \approx 1.0718$. Then:

$$k + 2cs = 0.168 + 0 = 0.168 < 1$$

All conditions of Theorem 3.4 are satisfied, ensuring a unique solution in $B_{0.64}$.
Remark 4.1. This example demonstrates the flexibility of our framework in handling nonlinearities beyond linear and memory terms. The quadratic term δx^2 introduces additional complexity in the estimation of both the local bound $M_\Phi(R)$ and the Lipschitz constant L_Φ. The successful application of our method to this case underscores its robustness for various hereditary systems with diverse nonlinear structures. Notably, the invariant radius R_0 is obtained by solving a quadratic inequality, which differs from the linear relations in previous examples, showcasing the adaptability of our approach to different functional forms.

5. CONCLUSION

This investigation has established a comprehensive framework for analyzing hereditary integral equations in b-metric spaces, addressing three significant limitations in existing literature. First, the derivation of explicit invariant radii R_0 via the condition $M_g + |\lambda| M_K M_\Phi(R_0)(b - a) \leq R_0$ successfully eliminates global boundedness assumptions, aligning more closely with physical realities where solutions remain locally constrained but may exhibit unbounded behavior in extended domains.

Second, our rigorous verification of operator continuity through the application of the dominated convergence theorem resolves a frequently overlooked technical gap in fixed-point applications to integral equations. This methodological contribution ensures mathematical completeness in operational analyses.

Third, while $c = 0$ proves sufficient for our symmetric memory terms, we have identified asymmetric hereditary systems (particularly those with mixed delay types) as promising candidates for future exploration of $c > 0$ regimes. The selection of $c = 0$ in our current work represents an optimal choice rather than a simplification, as our contractive estimate naturally yields $d(Tx, Ty) \leq k \cdot d(x, y)$, and since $d(x, y)$ consistently constitutes an element of $\max\{d(x, Tx), d(y, Ty), d(x, y)\}$, the more general condition is satisfied with the optimal constant $c = 0$.

Applications to delayed logistic models and RLC circuits have demonstrated the framework's practical efficacy. The parameterized invariant sets ($R_0 = 0.7$ and $R_0 = 0.64$) and contractive constants ($k \approx 0.113$ and $k \approx 0.168$) provide computationally verifiable criteria consistent with biological and electronic constraints, offering concrete validation of our theoretical approach.

Future research directions will focus on three primary areas: (i) Extension to distributed delays $\int_a^b \kappa(\theta) x(t - \theta) d\theta$ requiring measure-theoretic adaptations, (ii) Incorporation of stochastic perturbations using Ito-integral formulations, and (iii) Development of numerical schemes based on Picard iteration with error bounds proportional to $(k + 2cs)^n$. These advancements position b-metric fixed-point theory as a versatile tool for hereditary systems, effectively bridging abstract mathematical analysis with applied computational science.

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