

## TWO-DIMENSIONAL STRATIFIED FLOW OVER A DIPOLE AND DETERMINATION OF THE DIVIDED CURVE

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### Abstract:

In the study on two-dimensional stratified flow in a channel, Dube (2002) and Yih (1960) proposed that the two-dimensional stratified flow over a barrier in a channel can be investigated by taking a suitable combination of sources, sink and doublets in place of barrier. Trustrum (1964) and then Dube (2023 & 2025) considered, however independently, the problem of two-dimensional channel flow over a barrier by applying an Oseen-type approximation to the general flow and discussed the Long's hypothesis. In the study, she simply remarked that in the case of the flow over a dipole (in place of the barrier), if the axis of the dipole be parallel to the direction of the uniform stream at infinity then practically there is no far-upstream influence (or the influence is negligibly small). However, if the axis be perpendicular to the uniform stream at infinity, the far-upstream influence of the dipole is not negligible.

Drazin and Moore (1967) and then Dube (2025) used the technique of diffraction theory to obtain the solution of Long's linearized equation for the steady flow of an incompressible inviscid fluid of variable density over an obstacle in an infinite channel. They extended the consideration to the flow over a dipole placed at the bottom of the channel with its axis parallel to the direction of the uniform stream by using some transformations and remarked in the same way as used to describe the flow over any obstacle whose shape happens to coincide with a streamline. But it was pointed out that a grave difficulty arises in making the shape of the obstacle to coincide exactly with the streamline.

In the study of Drazin and Moore (1967) and then Dube (2025) on the flow past an obstacle it was indicated that blocking may arise depending on the pressure condition at infinity, the value of the Froude number and the size of the obstacle. Blocking may occur also on the case of the flow over a dipole depending on the pressure at infinity and the strength of the dipole. So, it is our aim to study here and find a relation between the pressure at infinity, Froude number and the strength of the dipole to be placed at the bottom of the channel with its axis parallel to it and directed against the uniform flow.

If the pseudo-elastic  $U_0$  at infinity on negative side (*at*  $x = -\infty$ ) be not large enough, then there is apparently a possibility that a layer of the stratified fluid in the lower region of the channel may not be able to cross that dipole. This will result in what may be called the clocking of the incoming fluid by the dipole. This leads rather to a contradiction to the remark by Trustrum (1954) that if the axis of the dipole be parallel to the channel wall then there is no possibility of blocking. So, we propose here to restudy the problem of stratified flow over a dipole placed at the bottom of an infinite channel with its axis parallel to the uniform flow at infinity. An attempt is also made, to find analytically the relation between the pressure condition at infinity (on the negative side) and the strength of the dipole for the non-occurrence of blocking.

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### Nomenclature:

$F$	:	Froude number
$F_0$	:	Ordinary Froude number

$\underline{q}$	:	Velocity vector
$\bar{q}$	:	Dimension of velocity of pseudo velocity
$\bar{F}$	:	External force other than gravitational force
$\bar{g}$	:	Acceleration due to gravity
$p$	:	Pressure
$\rho$	:	Density
$\rho_0$	:	Reference density
$U$	:	Horizontal velocity
$U_0$	:	Representative velocity
$d$	:	Reference length
$g$	:	Acceleration due to gravitation
$\psi$	:	Stream function
$\psi'$	:	Pseudo stream function
$\bar{k}$	:	Unit vector perpendicular to the plane of the motion
$\delta$	:	Variation of height of the streamline.

**Key Words:** Steady two-dimensional flow of a stably stratified, Incompressible inviscid fluid towards a sink, pseudo-flow, pseudo-stream function, torricellian vacuum.

### Governing Equation-boundary conditions and solutions:

We consider here a two-dimensional stratified flow over a dipole placed at the origin at the bottom of an infinite horizontal channel formed by  $y = 0$  and  $y = d$ . The axis of the dipole is horizontal and directed against the flow and the fluid of the purely dipole flow has constant density equal to that of the lowest stratum of the stratified fluid. The dipole flow affects the stratified fluid flow and at the same time it is affected by the stratified flow. Using the equation

$$\Psi = \int \left( \frac{\rho}{\rho_0} \right)^{-1/2} d\psi' \quad (1)$$

The physical flow is transformed into the pseudo-flow with uniform velocity at  $x = -\infty$ ; the stream function for the perturbed pseudo-flow in the channel in non-dimensional quantities is given by the equation

$$(\nabla^2 + \beta^2)\psi = \beta^2 y, \quad (2)$$

where  $\beta = \frac{1}{F}$ ,  $F$  being the Froude number and defined as  $F = \frac{U_0}{\sqrt{\alpha g d}}$ .

The boundary conditions for  $\psi$  are set as

$$\begin{aligned} \psi &\rightarrow y \text{ as } |x| \rightarrow \infty \\ \psi &= 1 \text{ on } y = 1 \forall x, \\ &= 0 \text{ on } y = 0 \text{ for } x = 0. \end{aligned} \quad (3)$$

Writing  $\Psi = \psi - y$ , then equation (2) becomes

$$(\nabla^2 + \beta^2)\Psi = 0, \quad (4)$$

With boundary conditions:

$$\begin{aligned} \Psi &\rightarrow 0 \text{ as } |x| \rightarrow \infty \\ \Psi &= 0 \text{ on } y = 1 \forall x, \\ &= 0 \text{ on } y = 0 \text{ for } x \neq 0. \end{aligned} \quad (5)$$

The function  $\Psi$  then defines the perturbation in the flow. In the absence of the dipole, the equation (4) admits the simple solution  $\Psi = 0$ , giving  $\psi = y$ , which represents the unperturbed uniform parallel pseudo-flow. In the perturbed pseudo-flow, the solution is, shown by Drazin and Moore (1967) and Dube (2025), found in two types depending on  $\beta$ . When  $\beta < \pi$ , the solution does not contain wavy terms and when  $\beta > \pi$ , the solution contains both wavy and non-wavy terms. Also  $\beta = 0$ , so solution can exist. This can also be seen from the following.

Assuming  $\Psi = \sum_{n=1}^{\infty} \Psi_n(x) \sin n\pi y$ , the equation (4) gives

$$\frac{d^2 \Psi_n}{dx^2} + (\beta^2 - n^2 \pi^2) \Psi_n = 0. \quad (6)$$

The solution of this equation depends on the sign of  $(\beta^2 - n^2 \pi^2)$ .

If  $0 < \beta < \pi$ , then  $(\beta^2 - n^2\pi^2)$  can never be positive and so the solution is of the type :

$$\Psi = A'_n \exp\left[-(\beta^2 - n^2\pi^2)^{1/2} x\right] + A_n \exp\left[-(n^2\pi^2 - \beta^2)^{1/2} x\right]$$

where  $A'_n$  and  $A_n$  are arbitrary constants; and if  $\beta$  is  $N\pi < \beta < (N+1)\pi$  for some positive integer  $N$ , then

$$\begin{aligned}(\beta^2 - n^2\pi^2) &> 0 \text{ for } n \leq N \\(\beta^2 - n^2\pi^2) &< 0 \text{ for } n \geq N+1\end{aligned}$$

and so for  $n \geq N+1$ , the solution will be of the above exponential type while for  $n \leq N$  the solution will be sinusoidal type

$$\Psi = A'_n \sin\left[(\beta^2 - n^2\pi^2)^{1/2} x\right] + A_n \cos\left[-(n^2\pi^2 - \beta^2)^{1/2} x\right]$$

which is stationary wave.

Thus when  $\beta > \pi$ , waves occur. To avoid the solution in which waves may occur, we shall restrict the consideration to the case  $\beta < \pi$ .

The solution shown above cannot, however, be accepted; for it cannot stratify all the necessary boundary conditions for the flow over a dipole. Thus the above method of solution fails, and hence we are to look for other method of solution.

Since the disturbance of the flow is caused by the dipole at the origin, therefore the solution must be that one having a dipole singularity at the origin. Drazin and Moore (1967) used the Dirac delta-function for the dipole singularity at the origin and accordingly the boundary condition on  $y = 0$  was set as  $\Psi = K\delta(x)$ .

Following Drazin and Moore (1967), the solution of the equation (4) and having a dipole singularity at the origin is, for  $\beta < \pi$ , written as

$$\Psi = -K\pi \sum_{n=1}^{\infty} \frac{n \sin n\pi y}{2^2} \exp\left[-x \sqrt{n^2\pi^2 - \beta^2}\right] \quad (7)$$

where  $K$  is some constant.

The perturbed pseudo-stream function  $\psi$  is the given by

$$\begin{aligned}\psi &= y + \Psi \\ &= y - K\pi \sum_{n=1}^{\infty} \frac{n \sin n\pi y}{2^{2n-1}} \exp\left[-x \left(n^2\pi^2 - \beta^2\right)^{1/2}\right] \quad (8)\end{aligned}$$

It may be noted that it is through the constant  $K$  that the dipole strength is to be involved.

The relationship between  $K$  and the dipole strength can be established simply by the arguing (after Drazin and Mooore) that the dipole flow in the neighbourhood of the origin is not affected by the stratification

and flow, i.e., the flow in the neighborhood of the origin (dipole singularity) is purely dipole flow.

Thus in the neighborhood of the origin, the equation (8) will reduce (by dropping the first term and also putting  $\beta = 0$ , i.e. in the dipole flow near the origin,  $\beta = 0$  as there is no stratification effect and also no term involving  $y$  alone can occur, the stream function simply satisfying Laplace equation) to

$$\begin{aligned}\psi &= -K\pi \sum_{n=1}^{\infty} \sin n\pi y \cdot \exp(-n\pi |x|), \\ &= -\frac{1}{2} K \frac{\sin \pi y}{\cosh \pi x - \cos \pi y}.\end{aligned}$$

The above result simplifies to (on expanding in powers of  $x$  and  $y$  and neglecting terms of  $O(x^3)$  and  $O(y^3)$  and higher orders)

$$\psi = -\frac{K}{\pi} \frac{y}{x^2 + y^2}. \quad (9)$$

Again, considering the purely dipole flow in the neighborhood of the origin, it is seen that the physical stream function  $\psi_1$  for the dipole flow may be taken as ( $\psi_1$  also satisfies the Laplace equation)

$$\psi_1 = -\mu_1 \frac{y}{x^2 + y^2}, \quad (x, y \text{ small}). \quad (10)$$

where  $\mu_1$  is the physical dipole strength.

Since the density of the fluid in the dipole flow is constant, the pseudo-stream function for the dipole flow is seen to be given by

$$\psi_2 = -\mu_1 \frac{y}{x^2 + y^2}, \quad (11)$$

where  $\psi_2 = \frac{\Psi_1}{Ud}$  and  $\mu = \frac{\mu_1}{U}$  are the non-dimensional dipole strength. (Here  $x, y$  are non-dimensional coordinates as used in the pseudo-flow of the stratified fluid).

Now, in the neighborhood of the origin, the equations (9) and (11) represent the same flow and so on, on comparison, one finds

$$K = \mu\pi . \quad (12)$$

This is the relation between the constant  $K$  and the non-dimensional dipole strength  $\mu$ . So, writing  $K$  in terms of  $\mu$ , the equation (8) now becomes

$$\psi = y - \mu\pi \sum_{n=1}^{\infty} \frac{n \sin n\pi y}{(n^2\pi^2 - \beta^2)^{1/2}} \exp\left[-x \sqrt{(n^2\pi^2 - \beta^2)^{1/2}}\right] \quad (13)$$

This gives the perturbed pseudo-stream function of the flow in the channel.

### Discussion:

As seen, for the flow not to contain waves,  $\beta$  should be less than  $\pi$  and also for the non-

occurrence of blocking  $\mu < \frac{1}{\pi}$  and  $p_\infty \geq \frac{\pi^2}{8} + \frac{\pi}{2} + F_0^{-2}$ .

The case  $\beta = 0 \Rightarrow \alpha = 0$  and so, in the case  $\beta = 0$ , the stratified fluid flow reduces to the homogeneous irrotational flow of constant density  $\rho = 1$  everywhere. In that case there is no distinction between the pseudo-stream function and the natural stream function. The flow is simply described by the stream function

$$\begin{aligned} \psi &= y - \mu\pi \sum_{n=1}^{\infty} \sin n\pi y \cdot \exp(-n\pi |x|) \\ \text{or } \psi &= y - \frac{1}{2} \mu\pi \frac{\sin \pi y}{\cosh \pi x - \cos \pi y} \quad (14) \end{aligned}$$

where  $\beta$  is not zero, but small enough, the flow pattern of the stratified fluid will not differ much from that of homogeneous fluid flow by equation (14).

Again  $\beta \neq 0$ ,

$$u = 1 - \mu\pi \sum_{n=1}^{\infty} \frac{n^2 \cos n\pi y}{(n^2\pi^2 - \beta^2)^{1/2}} \exp\left[-x \sqrt{(n^2\pi^2 - \beta^2)^{1/2}}\right]$$

$$\text{and } v = \mu\pi \sum_{n=1}^{\infty} n \sin n\pi y \exp\left[-|x| \left(n^2 \pi^2 - \beta^2\right)^{1/2}\right]$$

From these, it can be seen that the effect of  $\beta$  is to make the fluid move faster than in the corresponding homogeneous fluid.

The streamline  $\psi = 0$  demarcates the stratified fluid flow from the dipole flow region and so  $\psi > 0$  defines the stratified flow while  $\psi < 0$  defines the dipole flow.

**Conclusion:**

In the conclusion, we discuss the nature of change of flow with  $\beta$  and  $\mu$ . For the flow not contain waves,  $\beta$  should be less than  $\pi$  and also for the non-occurrence of blocking  $\mu \leq \frac{1}{\pi}$  and

$$P_{\infty} \geq \frac{\pi^2}{8} + \frac{\pi}{2} + F_0^{-2}$$

The case when  $\beta = 0$  implies  $\alpha = 0$  and so, in case  $\beta = 0$ , the stratified fluid flow reduces to the homogeneous irrotational flow of constant density  $\rho = 1$  everywhere. In that case, there is no distinction between the pseudo-stream function and the natural stream function. The flow is simply described by the stream function  $\psi$  as

$$\begin{aligned} \psi &= y - \mu\pi \sum_{n=1}^{\infty} \sin n\pi y \cdot \exp(-n\pi |x|) \\ &= y - \frac{1}{2} \mu\pi \frac{\sin \pi y}{\cosh \pi x - \cos \pi y} \end{aligned}$$

When  $\beta \neq 0$  but small enough, the flow pattern of the stratified fluid flow will not differ much from that of the homogeneous fluid flow given by the equation (36).

Again, for  $\beta \neq 0$ , we have

$$u = 1 - \mu\pi \sum_{n=1}^{\infty} \frac{n^2 \cos n\pi y}{(n^2 \pi^2 - \beta^2)^{1/2}} \exp\left[-|x| \left(n^2 \pi^2 - \beta^2\right)^{1/2}\right]$$

and

$$v = \mu\pi \sum_{n=1}^{\infty} n \sin n\pi y \cdot \exp\left[-|x| \left(n^2 \pi^2 - \beta^2\right)^{1/2}\right]$$

From these equations, the effect of  $\beta$  is to make the fluid move faster than that in the corresponding homogeneous fluid.

The streamline  $\psi = 0$  demarcates the stratified fluid flow from the dipole flow region and so  $\psi > 0$  defines the stratified flow while  $\psi < 0$  defines the dipole flow.

**Applications of the work:**

The study of stratified fluids also find applications in industries. The concept of solar pond and ocean thermal energy conversion (OTEC) may be mentioned. The intrusion of a heavy fluid into a lighter one occurs in the process of manufacturing glass. Idea of fluid flow in porous media is applicable to hydrology and is of vital interest to petroleum industries and paper industries. Axisymmetric flow of non-homogeneous fluids have also a bearing on such engineering devices as centrifuges and meteorological phenomena as tornados.

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