International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 20s, 2025
https://www.theaspd.com/ijes.php

The Chromatic Number of Fuzzy Cycle and Its Related Graphs

M.G. Karunambigai[1], Jessalet Ann Mathew[2*]

[1,2*]Department of Mathematics, Sri Vasavi College, Erode-638 316, Tamilnadu, India.
Email: karunsvc@yahoo.in

***Corresponding Author:** Jessalet Ann Mathew.
Email: jessaletannmathew@gmail.com

Abstract
This study focuses on determining the chromatic number of the middle graph, splitting graph, shadow graph, line graph, total graph and subdivision graph of the fuzzy cycle C_n. These results are obtained by using fuzzy colors based on the strength of edges incident on a vertex. Additionally, some key properties regarding the fuzzy coloring of the fuzzy graphs have been given.

Keywords: *Chromatic number, Fuzzy Path, Fuzzy Cycle, Middle graph, Splitting graph, Shadow graph, Line graph, Total graph, Subdivision graph.*

1. Introduction

Fuzzy graph coloring is an effective technique for solving many complex real-world problems. Since many practical situations can be represented as coloring problems, this area has become one of the most widely studied in graph theory. In particular, coloring fuzzy graphs plays a key role in resolving challenges in network systems [1]. Through the years, researchers have explored various approaches to coloring fuzzy graphs. The idea was first introduced by Susana Munoz et al. [2], who defined the chromatic number of a fuzzy graph as a fuzzy subset of its vertex set. Later, Eslahchi and Onagh [3] proposed a coloring method based on strong adjacency between vertices. After that, Sovan Samanta et al. [4] introduced a new approach that uses fuzzy colors determined by the strength of edges connected to each vertex.

Furthermore, we found the chromatic number of certain families of fuzzy graphs using fuzzy coloring [5] and is defined as follows:

(i) if two vertices are connected by a *strong* edge, then they either have different basic or fuzzy colors(if necessary), or one vertex can have a basic color and the other can have a fuzzy color corresponding to different basic color.

(ii) if two vertices are connected by a *weak* edge, then they either have same or different fuzzy colors, or one vertex can have a basic color and other can have a fuzzy color corresponding to the same basic color.

The minimum number of colors (basic or fuzzy) needed for a proper fuzzy coloring of G is called the chromatic number of G, is denoted by $\chi_f(G)$. We also derived some key properties related to this type of coloring. In this paper, we extend our study by determining the chromatic number of the middle graph, splitting graph, shadow graph, line graph, total graph and subdivision graph of fuzzy cycle C_n. These results are obtained using fuzzy colors based on the strength of edges incident to each vertex and we also proved important properties of these coloring methods.

The structure of this article is as follows : In Section 1, introduction to the fuzzy coloring of a fuzzy graph is given. In Section 2, some basic concepts in fuzzy coloring of fuzzy graphs that aid in the research have been reviewed. In Section 3, we determine the chromatic number of the middle graph, splitting graph, shadow graph, line graph, total graph and the subdivision graph of the fuzzy cycle $\boldsymbol{C_n}$. Section 4 presents the final conclusions of this study.

2. Preliminaries

This section begins with a review of some definitions from fuzzy graph theory and fuzzy coloring, which help in determining the chromatic numbers of some related graphs of the fuzzy cycle.

Definition 2.1. (*fuzzy graph* [6]) A fuzzy graph $G = (V, \sigma, \mu)$ is a pair of functions (σ, μ), where $\sigma : V \rightarrow [0,1]$ is a fuzzy subset of a non-empty set V, and $\mu : V \rightarrow [0,1]$ is a symmetric fuzzy relation on σ, such that the relation $\mu(v_i, v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$ is satisfied for all $v_i, v_j \in V$ and $(v_i, v_j) \in E \subset V \times V$.
Here, $\sigma(v_i)$ denote the degree of membership of the vertex v_i, and $\mu(v_i, v_j)$ denotes the degree of membership of the edge relation $e_{ij} = (v_i, v_j)$ on $V \times V$.
Note : In this paper, we denote $\sigma(v_i) \wedge \sigma(v_j) = min\{\sigma(v_i), \sigma(v_j)\}$, and $\sigma(v_i) \vee \sigma(v_j) = max\{\sigma(v_i), \sigma(v_j)\}$.

Definition 2.2. (*fuzzy path* [7]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph with underlying crisp graph G^*. A fuzzy path P_n in G is a sequence of distinct vertices $v_0, v_1, \dots, v_n$ such that $\mu(v_{i-1}, v_i) > 0, 1 \leq i \leq n$. Here $n \geq 1$ is called the length of the path P_n.

Definition 2.3. (*fuzzy cycle* [7]) A fuzzy path P_n in which $v_0 = v_n$ and $n \geq 3$, then P_n is called a fuzzy cycle C_n of length n.

Definition 2.4. (*strong edge, weak edge* [3]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph and an edge $e = (v_i, v_j) \in G$ is called **strong** if $\frac{1}{2}\{\sigma(v_i) \wedge \sigma(v_j)\} \leq \mu(v_i, v_j)$ and it is called **weak** otherwise.
Definition 2.5. (*strength of an edge* [3]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph and the **strength of an edge** $(v_i, v_j) \in G$ is denoted by,

$$I(v_i, v_j) = \frac{\mu(v_i, v_j)}{\sigma(v_i) \wedge \sigma(v_j)}.$$

Definition 2.6. (*strong fuzzy graph* [8]) A fuzzy graph $G = (V, \sigma, \mu)$ is called a strong fuzzy graph if each edge in G is a strong edge.

Definition 2 .7. (*middle graph* [9]) The middle graph $M_f(G)(V_M, \sigma_M, \mu_M)$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $M(G)(V_M, E_M)$, with the vertex set $V_M = V \cup V_{ij}$ where $V = \{v_i \mid v_i \in V\}$ and $V_{ij} = \{v_{ij} \,\forall\, (v_i, v_j) \in E\}$ and $\nu(M_f(G)) = n + 1 + n = 2n + 1$ and the edge set

$$E_M = \begin{cases} (v_{ij}, v_i), (v_{ij}, v_j) & \forall\, i \text{ and } j, \\ (v_{ij}, v_{rs}) & \text{if the edges } (v_i, v_j) \text{ and } (v_r, v_s) \text{ are adjacent in } G. \end{cases}$$

Then, $\sigma_M(v_i) = \sigma(v_i)$ *if* $v_i \in V, 0 \leq i \leq n$,
$\sigma_M(v_{ij}) = \mu(v_i, v_j)$ *if* $(v_i, v_j) \in E \,\forall\, i$ *and* j,
$\mu_M(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G,
and $\mu_M(v_i, v_{ij}) = \mu_M(v_j, v_{ij}) = \mu(v_i, v_j) \,\forall\, i$ *and* j.

Definition 2.8. (*splitting graph* [9]) The splitting graph $S_f(G)(V_S, \sigma_S, \mu_S)$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $S(G)(V_S, E_S)$, with the vertex set $V_S = V \cup V'$ where $V = \{v_i \mid v_i \in V\}$ and $V' = \{v_i' \,\forall\, v_i \in V\}$ and the edge set

$$E_S = \begin{cases} (v_i, v_j) & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G, \\ (v'_i, v_j) & \text{if } v_i' \in V' \text{and } v_j \in V \text{ that are adjacent to } v_i \in V. \end{cases}$$

Then, $\sigma_S(v_i) = \sigma_S(v_i') = \sigma(v_i)\ for\ v_i \in V\ and\ v_i' \in V'$,
$\mu_S(v_i, v_j) = \mu(v_i, v_j)\ if\ v_i\ and\ v_j\ are\ adjacent\ in\ V$,
and $\mu_S(v_i', v_j) = \sigma_S(v_i') \wedge \sigma_S(v_j)\ if\ v_i' \in V' and\ v_j \in V\ that\ are\ adjacent\ to\ v_i \in V$.

Definition 2.9. (*shadow graph* [9]) The shadow graph $D_{2_f}\ (G)(V_{D_2}, \sigma_{D_2}, \mu_{D_2})$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $D_2(G)(V_{D_2}, E_{D_2})$ is obtained by taking two copies of G namely G' and G'' with the vertex set $V_{D_2} = V' \cup V''$ where $V' = \{v_i'\ \forall\ v_i \in V\}$ and $V'' = \{v_i''\ \forall\ v_i \in V\}$ and the edge set

$$E_{D_2} = \begin{cases} (v_i', v_j'), (v_i'', v_j'') & if\ v_i\ and\ v_j\ are\ adjacent\ in\ G, \\ (v_i', v_j'') & if\ v_i' \in V' and\ v_j'' \in V'' that\ are\ adjacent\ to\ v_i'' \in V''. \end{cases}$$

Then, $\sigma_{D_2}(v') = \sigma_{D_2}(v'') = \sigma(v) for\ v \in V, v' \in V', v'' \in V''$,
$\mu_{D_2}(v_i'v_j') = \mu_{D_2}(v_i''v_j'') = \mu(v_iv_j)$, for $v_i', v_j' \in V', v_i'', v_j'' \in V'', v_i, v_j \in V$,
$\mu_{D_2}(v_i'v_j'') = \sigma(v_i') \wedge \sigma(v_j'')\ if\ v_i' \in V' and\ v_j'' \in V''$ that are adjacent to $v_i'' \in V''$.

Definition 2.10. (*line graph* [9]) The line graph $L_f(G)(V_L, \sigma_L, \mu_L)$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $L(G)(V_L, E_L)$, where the vertex set $V_L = \{v_{ij} \forall\ (v_i, v_j) \in E\}$ and edge set $E_L\ \{(v_{ij}, v_{rs})\ the\ edges\ (v_i, v_j)\ and\ (v_r, v_s)\ are\ adjacent\ in\ G\}$. Then, $\sigma_L(v_{ij}) = \mu(v_i, v_j)$ if $v_{ij} \in V_L$ and $\mu_L(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G.

Definition 2.11. (*total graph* [9]) The total graph $T_f(G)(V_T, \sigma_T, \mu_T)$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $T(G)(V_T, E_T)$, with the vertex set $V_T = V \cup V_{ij}$, where $V = \{v_i \mid v_i \in V\}$ and $V_{ij} = \{v_{ij}\ \forall\ (v_i, v_j) \in E\}$ and the edge set

$$E_T = \begin{cases} (v_i, v_j) & if\ v_i\ and\ v_j\ are\ adjacent\ in\ G, \\ (v_{ij}, v_i), (v_{ij}, v_j) & \forall\ i\ and\ j, \\ (v_{ij}, v_{rs}) & if\ the\ edges\ (v_i, v_j) and\ (v_r, v_s) are\ adjacent\ in\ G. \end{cases}$$

Then, $\sigma_T(v_i) = \sigma(v_i)\ if\ v_i \in V$,
$\sigma_T(v_{ij}) = \mu(v_i, v_j) if\ (v_i, v_j) \in E\ \forall\ i\ and\ j$,
$\mu_T(v_i, v_j) = \mu(v_i, v_j)\ if\ v_i\ and\ v_j\ are\ adjacent\ in\ G$,
$\mu_T(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G,
and $\mu_T(v_i, v_{ij}) = \mu_T(v_j, v_{ij}) = \mu(v_i, v_j) \forall\ i\ and\ j$.

Definition 2.12. (*subdivision graph* [9]) The subdivision graph $sd_f(G)(V_{sd}, \sigma_{sd}, \mu_{sd})$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $sd(G)(V_{sd}, E_{sd})$, with the vertex set $V_{sd} = V \cup V_{ij}$ where $V = \{v_i \mid v_i \in V\}$ and $V_{ij} = \{v_{ij}\ \forall\ (v_i, v_j) \in E\}$ and $E_{sd} = \{(v_{ij}, v_i), (v_{ij}, v_j) \forall\ i\ \&\ j\}$. Then, $\sigma_{sd}(v_{ij}) = \mu(v_i, v_j)$ if $(v_i, v_j) \in E\ \forall\ i\ \&\ j$ and $\mu_{sd}(v_i, v_{ij}) = \mu_{sd}(v_j, v_{ij}) = \mu(v_i, v_j)\ \forall\ i\ \&\ j$.

Definition 2.13. (*fuzzy coloring, proper fuzzy coloring* [2]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Fuzzy coloring is an assignment of basic or fuzzy colors to the vertices of a fuzzy graph G and it is proper,

(i) if two vertices are connected by a strong edge, then they either have different basic or fuzzy colors(if necessary), or one vertex can have a basic color and the other can have a fuzzy color corresponding to different basic color.

(ii) if two vertices are connected by a weak edge, then they either have same or different fuzzy colors, or one vertex can have a basic color and other can have a fuzzy color corresponding to the same basic color.

Definition 2.14. (*perfect fuzzy coloring* [2]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Perfect fuzzy coloring (optimal fuzzy coloring) is an assignment of minimum number of colors (basic or fuzzy) for a proper fuzzy coloring of G.

Definition 2.15. (*chromatic number* [2]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The minimum number of colors (basic or fuzzy) needed for a proper fuzzy coloring of G is called the chromatic number of G and is denoted by $\chi_f(G)$.

Lemma 2.1. [2] *Let P_n be a fuzzy path of length n. If all edges are weak in P_n, then* $\chi_f(P_n) = 1$.

Lemma 2.2. [2] *Let P_n be a fuzzy path of length n. If all the edges are strong in P_n, then* $\chi_f(P_n)$= 2.

Theorem 2.1. [2] *Let P_n be a fuzzy path of length n. If atleast one edge is strong in P_n, then* $\chi_f(P_n) = 2$.

Lemma 2.3. [2] *Let C_n be a fuzzy cycle of length n. If all edges are weak in C_n, then* $\chi_f(C_n) = 1$.

Lemma 2.4. [2] *Let C_n be a fuzzy cycle of length n. If all the edges are strong in C_n, then*

$$\chi_f(C_n) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Theorem 2.2. [2] *Let C_n be a fuzzy cycle of length n. If weak and strong edges are distributed in any sequence in C_n, then*

$$\chi_f(C_n) = \begin{cases} 3 & if\ \frac{n}{2}\ number\ of\ strong\ and\ weak\ edges\ are\ alternatively \\ & distributed\ in\ C_n,\ where\ n(\geq 6) is\ even, \\ 2 & otherwise. \end{cases}$$

Theorem 2.3. [9] $\chi_f(G) \geq \max\{\chi_f(G_i) : 1 \leq i \leq k\}$, *where* $G = G_1 \cup G_2 \cup \ldots \cup G_k$ *and* $G_i, 1 \leq i \leq k$ *are fuzzy graphs.*

Corollary 2.3.1.[9] $\chi_f(G) \geq \max\{\chi_f(G_i) : 1 \leq i \leq k\}$, *where* $G = G_1 \oplus G_2 \oplus \ldots \oplus G_k$ *and* $G_i, 1 \leq i \leq k$ *are edge disjoint fuzzy graphs.*

Theorem 2.4. [11] *The complete graph K_n has Hamiltonian decomposition for all n.*

Note.[2]. *i.e.,* $K_{2n+1} = \oplus\, nC_{2n+1}$ *and* $K_{2n} = C_{2n} \oplus nP_1$, where $\oplus$ denotes edge disjoint union.

3. The Chromatic Number of Some Related Graphs of Fuzzy Cycle

In this section, we determine the chromatic number of the middle graph $M_f(C_n)$, splitting graph $S_f(C_n)$, shadow graph $D_{2_f}(C_n)$, line graph $L_f(C_n)$, total graph $T_f(C_n)$ and the subdivision graph $sd_f(C_n)$ of the fuzzy cycle C_n.

In our previous work [2], Theorem 2.2 is stated as : *Let C_n be a fuzzy cycle of length n. If weak and strong edges are distributed in any sequence in C_n, then*

International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 20s, 2025
https://www.theaspd.com/ijes.php

$$\chi_f(C_n) = \begin{cases} 3 & if \ \frac{n}{2} \ number \ of \ strong \ and \ weak \ edges \ are \ alternatively \\ & distributed \ in \ C_n, \ where \ n(\geq 6) is \ even, \\ 2 & otherwise. \end{cases}$$

Upon further analysis and computational verification, it was found that this theorem holds under a more precise condition. Accordingly, we present the refined version below.

Theorem 3.1. *Let C_n be a fuzzy cycle of length n. If weak and strong edges are distributed in any sequence in C_n, then*

$$\chi_f(C_n) = \begin{cases} 3 & if \ \frac{n}{2} \ number \ of \ strong \ and \ weak \ edges \ are \ alternatively \\ & distributed \ in \ C_n, \ where \ n \geq 6, n \equiv 2 \ (mod \ \ 4), \\ 2 & otherwise. \end{cases}$$

Proof. Proof is similar as Theorem 2.2. (refer [2]).

3.1. The Chromatic Number of $M_f(C_n)$

Remark 3.1.1. Let $C_n : v_1 \, v_2 \dots v_n \, v_1$ be a fuzzy cycle of length n. Then $M_f(C_n) = C_n \oplus C_{2n}$ (by Theorem 2.4), where C_n is oriented as $C_n : v_1 \, v_2 \dots v_n \, v_1$ and C_{2n} is oriented as $C_{2n} : \ v_1 \, v_{12} \, v_2 \, v_{23} \dots v_n \, v_{n1} \, v_1$.

Lemma 3.1.1. *Let C_n be a fuzzy cycle of length n. Then $M_f(C_n)$ is a strong fuzzy graph.*

Proof. Let $C_n : v_1 \, v_2 \dots v_n \, v_1$ be a fuzzy cycle of length n. By the definition of middle graph, we have $\sigma_M(v_i) = \sigma(v_i) if \ v_i \in V, 1 \leq i \leq n,$

$\sigma_M(v_{ij}) = \mu(v_i, v_j) if \ (v_i, v_j) \in E \ \forall \ i \ \& \ j,$

$\mu_M(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G,

and $\mu_M(v_i, v_{ij}) = \mu_M(v_j, v_{ij}) = \mu(v_i, v_j) \ \forall \ i \ \& \ j.$

Then each edges of $M_f(C_n)$ satisfies the condition of a strong edge (by definition 2.4). Therefore, $M_f(C_n)$ is a strong fuzzy graph.

Theorem 3.1.1. *If $M_f(C_n)$ is a strong fuzzy graph, then $\chi_f(M_f(C_n)) = 3$.*

Proof. Let $C_n : \ v_1 \, v_2 \dots v_n \, v_1$ be a fuzzy cycle of length n. Then $M_f(C_n) = C_n \oplus C_{2n}$ (by Remark 3.1.1) and by Lemma 3.1.1, $M_f(C_n)$ is a strong fuzzy graph.

Case 1 : In C_n, if n is even.
Then by Lemma 2.4 we have, $\chi_f(C_n) = \chi_f(C_{2n}) = 2$. Then by Corollary 2.3.1,

$$\begin{aligned} \chi_f(M_f(C_n)) &= max\{\chi_f(C_n), \chi_f(C_{2n})\} + 1 \\ &= max\{2,2\}+1 \\ &= 3. \end{aligned}$$

Case 2 : In C_n, if n is odd.
Then by Lemma 2.4 we have, $\chi_f(C_n) = 3$ and $\chi_f(C_{2n}) = 2$. Then by Corollary 2.3.1,

$$\begin{aligned} \chi_f(M_f(C_n)) &= max\{\chi_f(C_n), \chi_f(C_{2n})\} \\ &= max\{3,2\} \\ &= 3. \end{aligned}$$

3.2. The Chromatic Number of $S_f(C_n)$

Remark 3.2.1. Let $C_n : v_1 v_2 \dots v_n v_1$ be a fuzzy cycle of length n. Then $S_f(C_n) = C_n \oplus nP_2$ (by Theorem 2.4), where C_n is oriented as $C_n : v_1 v_2 \dots v_n v_1$ and P_2 is oriented as $P_2 : v_{i-1} v_i' \ v_{i+1}$, $1 \leq i \leq n$ with $v_0 = v_n$ & $v_{n+1} = v_1$.

Lemma 3.2.1. *Let $C_n : v_1 v_2 \dots v_n v_1$ be a fuzzy cycle of length n. If all the edges are weak in C_n, then the edges of $C_n \in S_f(C_n)$ are weak while the edges of all paths $P_2 \in S_f(C_n)$ are strong.*

Proof. Proof follows from the definition of splitting graph and the definition of weak and strong edges.

Theorem 3.2.1. *Let C_n be a fuzzy cycle of length n. If all the edges are weak in C_n, then $\chi_f(S_f(C_n)) = 2$.*

Proof. Let $C_n : v_1 v_2 \dots v_n v_1$ be a fuzzy cycle of length n. Then $S_f(C_n) = C_n \oplus nP_2$ (by Remark 3.2.1) and by Lemma 3.2.1, the edges of $C_n \in S_f(C_n)$ are weak while the edges of all paths $P_2 \in S_f(C_n)$ are strong. Then by Lemma 2.3 we have, $\chi_f(C_n) = 1$ and by Lemma 2.2 we have, $\chi_f(P_2) = 2$. Therefore by Corollary 2.3.1,

$$\begin{aligned}\chi_f(S_f(C_n)) &= max\{\chi_f(C_n), \chi_f(P_2)\} \\ &= max\{1,2\} \\ &= 2.\end{aligned}$$

Lemma 3.2.2. *Let C_n be a fuzzy cycle of length n. If all the edges are strong in C_n, then $S_f(C_n)$ is a strong fuzzy graph.*

Proof. Proof follows from the definition of splitting graph and the definition of strong edge.

Theorem 3.2.2. *If $S_f(C_n)$ is a strong fuzzy graph, then*

$$\chi_f\big(S_f(C_n)\big) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Proof. Let $C_n : v_1 v_2 \dots v_n v_1$ be a fuzzy cycle of length n. Then $S_f(C_n) = C_n \oplus nP_2$ (by Remark 3.2.1) and by Lemma 3.2.2, $S_f(C_n)$ is a strong fuzzy graph.

Case 1 : In C_n, if n is even.

Then by Lemma 2.4 we have, $\chi_f(C_n) = 2$ and by Lemma 2.2 we have, $\chi_f(P_2) = 2$.
Therefore by Corollary 2.3.1,

$$\begin{aligned}\chi_f(S_f(C_n)) &= max\{\chi_f(C_n), \chi_f(P_2)\} \\ &= max\{2,2\} \\ &= 2.\end{aligned}$$

Case 2 : In C_n, if n is odd.

Then by Lemma 2.4 we have, $\chi_f(C_n) = 3$ and by Lemma 2.2 we have, $\chi_f(P_2) = 2$.
Therefore by Corollary 2.3.1, $\chi_f(S_f(C_n)) = 3$.

Lemma 3.2.3. *Let $C_n : v_1 v_2 \dots v_n v_1$ be a fuzzy cycle of length n. If weak and strong edges are distributed in any sequence in C_n, then the edges of $C_n \in S_f(C_n)$ are also weak and strong, which are distributed in any sequence in $S_f(C_n)$, while the edges of all paths $P_2 \in S_f(C_n)$ are strong.* (The proof will be similar as above lemma).

Theorem 3.2.3. *Let C_n be a fuzzy cycle of length n. If weak and strong edges are distributed in any sequence in C_n, then*

$$\chi_f\big(S_f(C_n)\big) = \begin{cases} 3 & if\ \frac{n}{2}\ number\ of\ strong\ and\ weak\ edges\ are\ alternatively \\ & distributed\ in\ C_n, where\ n \geq 6, n \equiv 2\ (mod\ \ 4), of\ S_f(C_n), \\ 2 & otherwise. \end{cases}$$

Proof. Let $C_n : v_1\, v_2 \dots v_n\, v_1$ be a fuzzy cycle of length n. Then $S_f(C_n) = C_n \oplus n\, P_2$ (by Remark 3.2.1) and by Lemma 3.2.3, the edges of $C_n \in S_f(C_n)$ are weak and strong, which are distributed in any sequence in $S_f(C_n)$, while the edges of all paths $P_2 \in S_f(C_n)$ are strong.

Case 1 : Suppose $\frac{n}{2}$ number of strong and weak edges are alternatively distributed in C_n, where $n \geq 6, n \equiv 2\ (mod\ \ 4)$, of $S_f(C_n)$. Then by Theorem 3.1 we have, $\chi_f(C_n) = 3$ and by Lemma 2.4 we have, $\chi_f(P_2) = 2$. Therefore by Corollary 2.3.1,

$$\begin{aligned} \chi_f(S_f(C_n)) &= max\{\chi_f(C_n), \chi_f(P_2)\} \\ &= max\{3,2\} \\ &= 3. \end{aligned}$$

Case 2 : Suppose $\frac{n}{2}$ number of strong and weak edges are alternatively distributed in C_n, where $n \geq 8, n \equiv 0\ (mod\ \ 4)$, of $S_f(C_n)$.
Then by Theorem 3.1 we have, $\chi_f(C_n) = 2$ and by Lemma 2.4 we have, $\chi_f(P_2) = 2$. Therefore by Corollary 2.3.1, $\chi_f(S_f(C_n)) = 2$.

Case 3 : Suppose $\left\lfloor\frac{n}{2}\right\rfloor$ number of strong and weak edges are alternatively distributed in $C_n, n\ (\geq 3)$ is odd of $S_f(C_n)$.
Then by Theorem 3.1 we have, $\chi_f(C_n) = 2$ and by Lemma 2.4 we have, $\chi_f(P_2) = 2$. Therefore by Corollary 2.3.1, $\chi_f(S_f(C_n)) = 2$.

Case 4 : Suppose weak and strong edges are distributed in any sequence (except alternative distribution) in C_n of $S_f(C_n)$.
Then by Theorem 3.1 we have, $\chi_f(C_n) = 2$ and by Lemma 2.4 we have, $\chi_f(P_2) = 2$. Therefore by Corollary 2.3.1, $\chi_f(S_f(C_n)) = 2$.

3.3. The Chromatic Number of $D_{2_f}(C_n)$

Remark 3.3.1. Let $C_n : v_1\, v_2 \dots v_n\, v_1$ be a fuzzy cycle of length n. Then $D_{2_f}(C_n) = \oplus\, 2C_n \oplus nP_2$ (by Theorem 2.4), where the cycles C_n are oriented as $C_n: v_1'\, v_2' \dots v_n'\, v_1'$ & $C_n : v_1''\, v_2'' \dots v_n''\, v_1''$ and P_2 is oriented as $P_2 : v_{i-1}'\, v_i''\, v_{i+1}', 1 \leq i \leq n$.

Lemma 3.3.1. *Let $C_n :\ v_1\, v_2 \dots v_n\, v_1$ be a fuzzy cycle of length n. If all the edges are weak in C_n, then the edges $(v_i', v_{i+1}'), 1 \leq i \leq n$ and $(v_i'', v_{i+1}''), 1 \leq i \leq n$ are weak in $D_{2_f}(C_n)$, while the edges of all paths $P_2 \in D_{2_f}(C_n)$ are strong.*

Proof. Proof follows from the definition of shadow graph and the definition of weak and strong edges.

Theorem 3.3.1. *Let C_n be a fuzzy cycle of length n. If all the edges are weak in C_n, then $\chi_f(D_{2_f}(C_n)) = 2$.*

Proof. Let $C_n : v_1\, v_2 \dots v_n\, v_1$ be a fuzzy cycle of length n. Then $D_{2_f}(C_n) = \oplus\, 2C_n \oplus nP_2$ (by Remark 3.3.1) and by Lemma 3.3.1, the edges $(v_i', v_{i+1}'), 1 \leq i \leq n$ and $(v_i'', v_{i+1}''), 1 \leq i \leq n$ are weak in $D_{2_f}(C_n)$, while the edges of all paths $P_2 \in D_{2_f}(C_n)$ are strong. Then by Lemma 2.3 we have, $\chi_f(C_n) = 1$ and by Lemma 2.2 we have, $\chi_f(P_2) = 2$. Therefore by Corollary 2.3.1, $\chi_f(D_{2_f}(C_n)) = 2$.

Lemma 3.3.2. *Let C_n be a fuzzy cycle of length n. If all the edges are strong in C_n, then $D_{2_f}(C_n)$ is a strong fuzzy graph.*

Proof. Proof follows from the definition of fuzzy shadow graph and the definition of strong edge.

Theorem 3.3.2. *If $D_{2_f}(C_n)$ is a strong fuzzy graph, then*

$$\chi_f(D_{2_f}(C_n)) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Proof. Let $C_n : v_1\, v_2 \dots v_n\, v_1$ be a fuzzy cycle of length n. Then $D_{2_f}(C_n) = \oplus\, 2C_n \oplus nP_2$ and by Lemma 3.3.2, all edges are strong in $D_{2_f}(C_n)$.

Case 1 : In C_n, if n is even.
Then by Lemma 2.4 we have, $\chi_f(C_n) = 2$ and by Lemma 2.2 we have, $\chi_f(P_2) = 2$.
Therefore by Corollary 2.3.1,

$$\begin{aligned} \chi_f(D_{2_f}(C_n)) &= max\{\chi_f(C_n), \chi_f(P_2)\} \\ &= max\{2,2\} \\ &= 2. \end{aligned}$$

Case 2 : In C_n, if n is odd.
Then by Lemma 2.4 we have, $\chi_f(C_n) = 3$ and by Lemma 2.2 we have, $\chi_f(P_2) = 2$.
Therefore by Corollary 2.3.1, $\chi_f(D_{2_f}(C_n)) = 3$.

Lemma 3.3.3. *Let $C_n :\ v_1\, v_2 \dots v_n\, v_1$ be a fuzzy cycle of length n. If weak and strong edges are distributed in any sequence in C_n, then the edges $(v_i', v_{i+1}'), 1 \le i \le n-1$ and $(v_i'', v_{i+1}''), 1 \le i \le n-1$ are weak and strong, which are distributed in any sequence in $D_{2_f}(C_n)$, while the edges of all paths $P_2 \in D_{2_f}(C_n)$ are strong.* (The proof will be similar as above lemma).

Theorem 3.3.3. *Let C_n be a fuzzy cycle of length n. If weak and strong edges are distributed in any sequence in C_n, then*

$$\chi_f(D_{2_f}(C_n)) = \begin{cases} 4 & if\ strong\ and\ weak\ edges\ are\ alternatively\ distributed\ in\ C_4, \\ 4 & if\ \frac{n}{2}\ number\ of\ strong\ and\ weak\ edges\ are\ alternatively \\ & distributed\ in\ C_n, where\ n \ge 6, n \equiv 2\ (mod\ \ 4), of\ D_{2_f}(C_n), \\ 3 & otherwise. \end{cases}$$

Proof. Let $C_n : v_1\, v_2 \dots v_n\, v_1$ be a fuzzy cycle of length n. Then $D_{2_f}(C_n) = \oplus\, 2C_n \oplus nP_2$ (by Remark 3.3.1) and by Lemma 3.3.3, the edges $(v_i', v_{i+1}'), 1 \le i \le n-1$ and $(v_i'', v_{i+1}''), 1 \le i \le\ n-1$ are weak and strong, which are distributed in any sequence in $D_{2_f}(C_n)$, while the edges of all paths $P_2 \in D_{2_f}(C_n)$ are strong.

Case 1 : Suppose strong and weak edges are alternatively distributed in C_4.
Then by Theorem 3.1 we have, $\chi_f(C_4) = 2$ and by Lemma 2.4 we have, $\chi_f(P_2) = 2$.
Therefore by Corollary 2.3.1,

$$\begin{aligned} \chi_f(D_{2_f}(C_4)) &= max\{\chi_f(C_4), \chi_f(P_2)\} + 2 \\ &= max\{2,2\} + 2 \\ &= 4. \end{aligned}$$

Case 2 : Suppose $\frac{n}{2}$ number of strong and weak edges are alternatively distributed in C_n, where $n \geq 6, n \equiv 2\ (mod\ \ 4)$, of $D_{2_f}(C_n)$.
Then by Theorem 3.1 we have, $\chi_f(C_n) = 3$ and by Lemma 2.4 we have, $\chi_f(P_2) = 2$.
Therefore by Corollary 2.3.1,

$$\chi_f(D_{2_f}(C_n)) = max\{\chi_f(C_n), \chi_f(P_2)\} + 1$$
$$= max\{3,2\} + 1$$
$$= 4.$$

Case 3 : Suppose $\frac{n}{2}$ number of strong and weak edges are alternatively distributed in C_n, where $n \geq 8, n \equiv 0\ (mod\ \ 4)$, of $D_{2_f}(C_n)$.
Then by Theorem 3.1 we have, $\chi_f(C_n) = 2$ and by Lemma 2.4 we have, $\chi_f(P_2) = 2$.
Therefore by Corollary 2.3.1,

$$\chi_f(D_{2_f}(C_n)) = max\{\chi_f(C_n), \chi_f(P_2)\} + 1$$
$$= max\{2,2\} + 1$$
$$= 3.$$

Case 4 : Suppose $\left\lfloor\frac{n}{2}\right\rfloor$ number of strong and weak edges are alternatively distributed in $C_n, n\ (\geq 3)$ is odd of $D_{2_f}(C_n)$.
Then by Theorem 3.1 we have, $\chi_f(C_n) = 2$ and by Lemma 2.4 we have, $\chi_f(P_2) = 2$. Therefore by Corollary 2.3.1,

$$\chi_f(D_{2_f}(C_n)) = max\{\chi_f(C_n), \chi_f(P_2)\} + 1$$
$$= max\{2,2\} + 1$$
$$= 3.$$

Case 5 : Suppose weak and strong edges are distributed in any sequence (except alternative distribution) in C_n of $D_{2_f}(C_n)$.
Then by Theorem 3.1 we have, $\chi_f(C_n) = 2$ and by Lemma 2.4 we have, $\chi_f(P_2) = 2$.
Therefore by Corollary 2.3.1, $\chi_f(D_{2_f}(C_n)) = 3$.

3.4. The Chromatic Number of $L_f(C_n)$

Lemma 3.4.1. *Let C_n be a fuzzy cycle of length n. Then $L_f(C_n)$, is a strong fuzzy graph.*

Proof. Proof follows from the definition of fuzzy line graph and the definition of strong edge.

Theorem 3.4.1. *If $L_f(C_n)$ is a strong fuzzy graph, then*

$$\chi_f\big(L_f(C_n)\big) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Proof. Let $C_n: v_1 v_2 \dots v_n\, v_1$ be a fuzzy cycle of length n. Since $L_f(C_n) \cong C_n$, then by Lemma 2.4, the results follows.

3.5. The Chromatic Number of $T_f(C_n)$

Remark 3.5.1. Let $C_n\ :\ v_1\, v_2 \dots v_n\, v_1$ be a fuzzy cycle of length n. Then $T_f(C_n) = C_n \oplus L_f(C_n) \oplus nP_2$ (by Theorem 2.4), where C_n is oriented as $C_n\ : v_1 v_2 \dots v_n v_1$, $L_f(C_n)$ is oriented as $L_f(C_n)$: $v_{12} v_{23} \dots\ v_{n1}$ and P_2 is oriented as $P_2\ :\ v_i\ v_{ii+1} v_{i+1}$, $1 \leq i \leq n$ with $v_{n+1} =\ v_1$.

Lemma 3.5.1. *Let C_n be a fuzzy cycle of length n. If all the edges are weak in C_n, then the edges of $C_n \in T_f(C_n)$ are weak while the edges of $L_f(C_n) \in T_f(C_n)$ and the edges of all paths $P_2 \in T_f(C_n)$ are strong.*

Proof. Proof follows from the definition of total graph and the definition of weak and strong edges.

Theorem 3.5.1. *Let C_n be a fuzzy cycle of length n. If all the edges are weak in C_n, then*

$$\chi_f(T_f(C_n)) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Proof. Let $C_n : v_1\, v_2 \ldots v_n\, v_1$ be a fuzzy cycle of length n. Then $T_f(C_n) = C_n \oplus L_f(C_n) \oplus nP_2$ (by Theorem 2.4) and by Lemma 3.5.1, the edges of $C_n \in T_f(C_n)$ are weak while the edges of $L_f(C_n) \in T_f(C_n)$ and the edges of all paths $P_2 \in T_f(C_n)$ are strong. Then by Lemma 2.2 we have, $\chi_f(P_2) = 2$, by Lemma 2.3 we have, $\chi_f(C_n) = 1$ and by Theorem 3.4.1 we have,

$$\chi_f(L_f(C_n)) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Case 1 : In $L_f(C_n)$, if n is even.
Then by Corollary 2.3.1,

$$\begin{aligned} \chi_f(T_f(C_n)) &= max\{\chi_f(C_n), \chi_f(L_f(C_n)), \chi_f(P_2)\} \\ &= max\{1,2,2\} \\ &= 2. \end{aligned}$$

Case 2 : In $L_f(C_n)$, if n is odd.
Then by Corollary 2.3.1,

$$\begin{aligned} \chi_f(T_f(C_n)) &= max\{\chi_f(C_n), \chi_f(L_f(C_n)), \chi_f(P_2)\} \\ &= max\{1,3,2\} \\ &= 3. \end{aligned}$$

Lemma 3.5.2. *Let C_n be a fuzzy path of length n. If all the edges are strong in C_n, then $T_f(C_n)$ is a strong fuzzy graph.*

Proof. Proof follows from the definition of total graph and the definition of strong edge.

Theorem 3.5.2. *If $T_f(C_n)$ is a strong fuzzy graph, then*

$$\chi_f(T_f(C_n)) = \begin{cases} 3 & if\ n = 3, \\ 4 & otherwise. \end{cases}$$

Proof. Let $C_n : v_1\, v_2 \ldots v_n\, v_1$ be a fuzzy cycle of length n. Then $T_f(C_n) = C_n \oplus L_f(C_n) \oplus nP_2$ (by Remark 3.5.1) and by Lemma 3.5.2, all edges are strong in $T_f(C_n)$. Then by Lemma 2.2 we have, $\chi_f(P_2) = 2$ and by Lemma 2.4 & Theorem 3.4.1 we have,

$$\chi_f(C_n) = \chi_f(L_f(C_n)) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Case 1 : In C_n, if $n = 3$.
Then by Corollary 2.3.1,

$$\begin{aligned} \chi_f(T_f(C_3)) &= max\{\chi_f(C_3), \chi_f(L_f(C_3)), \chi_f(P_2)\} \\ &= max\{3,3,2\} \\ &= 3. \end{aligned}$$

Case 2 : In C_n, if n is odd but $n \neq 3$.
Then by Corollary 2.3.1,

$$\begin{aligned}\chi_f(T_f(C_n)) &= max\{\chi_f(C_n), \chi_f(L_f(C_n)), \chi_f(P_2)\} + 1\\ &= max\{3,3,2\} + 1\\ &= 4.\end{aligned}$$

Case 3 : In C_n, if n is even.
Then by Corollary 2.3.1,

$$\begin{aligned}\chi_f(T_f(C_n)) &= max\{\chi_f(C_n), \chi_f(L_f(C_n)), \chi_f(P_2)\} + 2\\ &= max\{2,2,2\} + 2\\ &= 4.\end{aligned}$$

Lemma 3.5.3. *Let $C_n : v_1\ v_2 \dots v_n\ v_1$ be a fuzzy cycle of length n. If weak and strong edges are distributed in any sequence in C_n, then the edges of $C_n \in T_f(C_n)$ are weak and strong, which are distributed in any sequence in $T_f(C_n)$, while the edges of $L_f(C_n) \in T_f(C_n)$ and the edges of all paths $P_2 \in T_f(C_n)$ are strong.* (The proof will be similar as above lemma).

Theorem 3.5.3. *Let C_n be a fuzzy cycle of length n. If weak and strong edges are distributed in any sequence in C_n, then*

$$\chi_f(T_f(C_n)) = \begin{cases} 3 & if\ strong\ and\ weak\ edges\ are\ alternatively\ distributed\ in\ C_3,\\ 3 & if\ \frac{n}{2}\ number\ of\ strong\ and\ weak\ edges\ are\ alternatively\\ & distributed\ in\ C_n, where\ n \geq 6, n \equiv 2\ (mod\ \ 4), of\ T_f(C_n),\\ 4 & otherwise.\end{cases}$$

Proof. Let $C_n :\ v_1\ v_2 \dots v_n\ v_1$ be a fuzzy cycle of length n. Then $T_f(C_n) = C_n \oplus L_f(C_n) \oplus nP_2$ (by Remark 3.5.1) and by Lemma 3.5.3, the edges of $C_n \in T_f(C_n)$ are weak and strong, which are distributed in any sequence in $T_f(C_n)$, while the edges of $L_f(C_n) \in T_f(C_n)$ and the edges of all paths $P_2 \in T_f(C_n)$ are strong. Then by Lemma 2.2 we have, $\chi_f(P_2) = 2$, by Theorem 3.1 we have,

$$\chi_f(C_n) = \begin{cases} 3 & if\ \frac{n}{2}\ number\ of\ strong\ and\ weak\ edges\ are\ alternatively\\ & distributed\ in\ C_n, where\ n \geq 6, n \equiv 2\ (mod\ \ 4),\\ 2 & otherwise.\end{cases}$$

and by Theorem 3.4.1 we have,

$$\chi_f(L_f(C_n)) = \begin{cases} 2 & if\ n\ is\ even,\\ 3 & if\ n\ is\ odd.\end{cases}$$

Case 1 : Suppose strong and weak edges are alternatively distributed in C_3.
Then by Corollary 2.3.1,

$$\begin{aligned}\chi_f(T_f(C_3)) &= max\{\chi_f(C_3), \chi_f(L_f(C_3)), \chi_f(P_2)\}\\ &= max\{2,3,2\}\\ &= 3.\end{aligned}$$

Case 2 : Suppose $\frac{n}{2}$ number of strong and weak edges are alternatively distributed in C_n, where $n \geq 6, n \equiv 2\ (mod\ \ 4), of\ T_f(C_n)$.
Then by Corollary 2.3.1,

$$\begin{aligned}\chi_f(T_f(C_n)) &= max\{\chi_f(C_n), \chi_f(L_f(C_n)), \chi_f(P_2)\} + 1\\ &= max\{3,2,2\} + 1\\ &= 4.\end{aligned}$$

Case 3 : Suppose $\frac{n}{2}$ number of strong and weak edges are alternatively distributed in C_n, where $n \geq 8, n \equiv 0 \ (mod \ \ 4), of \ T_f(C_n)$.
Then by Corollary 2.3.1,

$$\begin{aligned}\chi_f(T_f(C_n)) &= max\{\chi_f(C_n), \chi_f(L_f(C_n)), \chi_f(P_2)\} + 1\\ &= max\{2,2,2\} + 1\\ &= 3.\end{aligned}$$

Case 4 : Suppose strong and weak edges are distributed in any sequence in C_n, n $(\geq \ 3)$is odd, $of \ T_f(C_n)$.
Then by Corollary 2.3.1,

$$\begin{aligned}\chi_f(T_f(C_n)) &= max\{\chi_f(C_n), \chi_f(L_f(C_n)), \chi_f(P_2)\} + 1\\ &= max\{2,3,2\} + 1\\ &= 4.\end{aligned}$$

Case 5 : Suppose weak and strong edges are distributed in any sequence in C_n, n $(\geq \ 4)$ is even, $of \ T_f(C_n)$. (except all the above cases).
Then by Corollary 2.3.1,

$$\begin{aligned}\chi_f(T_f(C_n)) &= max\{\chi_f(C_n), \chi_f(L_f(C_n)), \chi_f(P_2)\} + 1\\ &= max\{2,2,2\} + 2\\ &= 4.\end{aligned}$$

Note : In C_n, if $n = 2$, $\chi_f(T_f(C_2)) = 2$.

3.6. The Chromatic Number of $sd_f(C_n)$

Lemma 3.6.1. *Let C_n be a fuzzy cycle of length n. Then $sd_f(C_n)$ is a strong fuzzy graph.*
Proof. Proof follows from the definition of fuzzy subdivision graph and the definition of strong edge.

Theorem 3.6.1. *If $sd_f(C_n)$ is a strong fuzzy graph, then $\chi_f(sd_f(C_n)) = 2$.*
Proof. Let $C_n : v_1 v_2 \dots v_n v_1$ be a fuzzy cycle of length n. Since $sd_f(C_n) \cong C_{2n}$, then by Lemma 2.2, $\chi_f(sd_f(C_n)) = 2$.

4. Conclusion

Fuzzy coloring serves as a significant extension of classical graph coloring into the domain of fuzzy graph theory, enabling more flexible modeling and effective problem-solving in systems characterized by uncertainty and imprecise relationships. The chromatic number of a fuzzy graph provides a powerful tool for addressing real-world problems with greater accuracy and flexibility.

In this paper, we determined the chromatic numbers of the middle graph, splitting graph, shadow graph, line graph, total graph, and the subdivision graph of the fuzzy cycle C_n, by using fuzzy coloring based on the strength of the edges incident on each vertex.

5. References

[1] Kannan, M., Sathiragavan, M., Nivetha, P., Sankar, K., and Gurjar, J., 2024, "Graph Coloring Techniques in Scheduling and Resource Allocation," *J. Nonlinear Anal. Optim.*, **15**(2).

[2] Karunambigai, M. G., and Mathew, J. A., 2024, "The Chromatic Number of Certain Families of Fuzzy Graphs," *Cuest. Fisioter.*, **53**(3), pp. 5153–5168.

[3] Munoz, S., Ortuno, M. T., Ramirez, J., and Yanez, J., 2005, "Coloring Fuzzy Graphs," *Omega*, **33**(3), pp. 211–221.

[4] Eslahchi, C., and Onagh, B. N., 2006, "Vertex-Strength of Fuzzy Graphs," *Int. J. Math. Math. Sci.*, **2006**(1), p. 043614.

[5] Samanta, S., Pramanik, T., and Pal, M., 2016, "Fuzzy Colouring of Fuzzy Graphs," *Afr. Mat.*, **27**, pp. 37–50.

[6] Mathew, S., Mordeson, J. N., and Malik, D. S., 2018, *Fuzzy Graph Theory*, Springer International Publishing, Berlin, Germany.

[7] Pal, M., Samanta, S., and Ghorai, G., 2020, "Fuzzy Cut Vertices and Fuzzy Trees," in *Modern Trends in Fuzzy Graph Theory*, Springer, Berlin, Ch. 3, pp. 115–124.

[8] Chithra, K. P., and Pilakkat, R., 2019, "Strength of Cartesian Product of Certain Strong Fuzzy Graphs," *Palest. J. Math.*, **8**(1).

[9] Karunambigai, M. G., and Mathew, J. A.,2025, "The Chromatic Number of Fuzzy Paths and Its Related Graphs," *Int. J. Environ. Sci.*, **11**(18s), pp. 1858-1869.

[10] Gong, Z., and Zhang, J., 2022, "Chromatic Number of Fuzzy Graphs: Operations, Fuzzy Graph Coloring, and Applications," *Axioms*, **11**(12), p. 697.

[11] Karunambigai, M. G., and Muthusamy, A., 2006, "On Resolvable Multipartite G-Designs II," *Graphs and Combinatorics*, **22**(1), pp. 59–67.