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The Role Of Edge -Vertex Interactions In Irredundant Sets Of Middle Graph

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Abstract
In this paper, we investigate the irredundant set and irredundant number of middle graphs of Path, Cycle and Star. The minimum cardinality of a maximal irredundant set in G is called the irredundance number of G and is denoted by $ir(G)$. The irredundant number, $ir(G) = min\{|S|: S\ is\ a\ maximal\ irredundant\ set\}$.
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INTRODUCTION

A set of vertices $S \subseteq V(G)$ is an irredundant set if for every vertex $v \in S, N[S-\{v\}] \neq N[S]$. The private neighborhood $pn[v,S]$ of $v \in S$ is defined by $pn[v,S] = N[v] - N[S-\{v\}]$. If S is an irredundant set of vertices in a graph G, then for each $v \in S$, the set $pn[v,S]$ is nonempty. If $pn[v,S]$ is empty, then S is a redundant set. A maximal irredundant set is an irredundant set that cannot be expanded to another irredundant set by addition of any vertex in the graph. The minimum cardinality of a maximal irredundant set in G is called the irredundance number of G and is denoted by $ir(G)$. The irredundant number, $\mathrm{ir(G) = min\{|S|: S\ is\ a\ maximal\ irredundant\ set\}}$. The number of neighbors of a vertex is called its degree, $d(v) = |N(v)|$. Irredundance concepts in graphs was defined by Cockayne.

Definition : The middle graph of a connected graph G denoted by $M(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.[8]

Middle graph of known graphs and their irredundant number

Theorem 1: Let $v_1 v_2 \cdots v_n$ be the path P_n and $u_i = v_i v_{i+1}, 1 \le i \le n-1$. Then, in $M(P_n), S = \{u_1, u_3, u_5, \cdots, u_{n-1}\}$ is a minimum irredundant set for $n \ge 3$.

Proof. Given, path P_n is $v_1 v_2 \cdots v_n$ and $u_i = v_i v_{i+1}, 1 \le i \le n-1$. Then $V(M(P_n)) = \{v_i, u_j / 1 \le i \le n, 1 \le j \le n-1\}$. In $M(P_n)$, v_1 is adjacent to u_1, v_n is adjacent to u_{n-1}, v_i is adjacent to u_{i-1} and u_i for $2 \le i \le n-1$, u_1 is adjacent to v_1, v_2 and u_2, u_{n-1} is adjacent to v_{n-1}, v_n and u_{n-2} and u_i is adjacent to u_{i-1}, u_{i+1}, v_i and v_{i+1}, for $2 \le i \le n-2$. Clearly, $|V(M(P_n))| = 2n-1$ and $|E(M(P_n))| = 3n-4$.

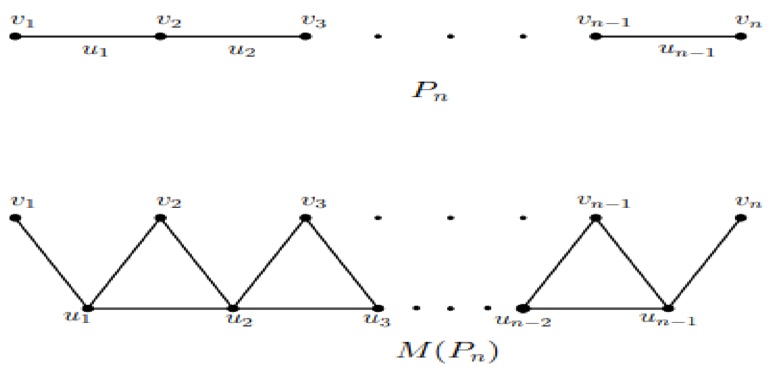

Fig. 1

Here, $N[u_1]=\{v_1,v_2,u_1,u_2\}, N[u_3]=\{v_3,v_4,u_2,u_3,u_4\}, N[u_5]=\{v_5,v_6,u_4,u_5,u_6\}$. Proceeding like this we get $N[u_i]=\{v_i,v_{i+1},u_{i-1},u_i,u_{i+1}\}$ where $i\equiv 1(mod\ 2)$ and $N[u_{n-1}]=\{v_{n-1},v_{n+1},u_{n-2},u_{n-1}\}$. Here, $N[u_1]\neq N[u_3]\neq\cdots\neq N[u_{n-1}]$. Hence, we obtain a set $S=\{u_1,u_3,\cdots,u_{n-1}\}$ where $N[S]=V(M(P_n))$. Clearly, $pn[u_1,S]=\{v_1,v_2\}, pn[u_3,S]=\{v_3,v_4\},\cdots, pn[u_{n-1}]=\{v_{n-1},v_n\}$. Clearly, for each vertex $u_i\in S$, $pn[u_i,S]\neq\phi$. Therefore, S is an irredundant set. Now, we show that S is a minimum irredundant set for $M(P_n)$ that is to show that S is a maximal irredundant set with a minimum cardinality. Suppose, S is not a maximal irredundant set. Then there exists a vertex $w\in V-S$, where w can be either v_i or u_j, for which $S'=S\cup\{w\}$ is irredundant, $1\le i\le n$ and $j\in\{2,4,\cdots,n-2\}$. In particular that $pn[w,S']\neq\phi$, for all $w\in S'$. If $w=v_i, 1\le i\le n$, then $pn[w,S']=pn[v_i,S']=N[v_i]-N[S'-\{v_i\}]=\{u_{i-1},u_i\}-V(M(P_n))=\phi$. Similarly, for $w=u_j, j\in\{2,4,\cdots,n-2\}, pn[w,S']=\phi$. Hence, S' is a redundant set which is contradiction to S' is an irredundant set. Thus, S is a maximal irredundant set with a minimum cardinality. Hence, $S=\{u_1,u_3,\cdots,u_{n-1}\}$ is a minimum irredundant set for $M(P_n)$ where n is even. Similarly, if n is odd, then $S=\{u_1,u_3,\cdots,u_{n-2},u_{n-1}\}$ is a minimum irredundant set for $M(P_n)$. Therefore, $S=\{u_1,u_3,u_5,\cdots,u_{n-1}\}$ is a minimum irredundant set for $n\ge 3$.

Theorem 2: For the graph $M(P_n)$ where $n\ge 3$, $ir(M(P_n)=\lceil\frac{n}{2}\rceil$.

Proof. By Theorem 1, if n is even then $S=\{u_1,u_3,\cdots,u_{n-1}\}$ is a minimum irredundant set for $M(P_n)$ with $|S|=\frac{n}{2}$ and if n is odd, then $S=\{u_1,u_3,\cdots,u_{n-2},u_{n-1}\}$ is a minimum irredundant set for $M(P_n)$ with $|S|=\frac{n-1}{2}+1$. This shows that $ir(M(P_n))=\lceil\frac{n}{2}\rceil$.

Example: Consider the graph $M(P_6)$ given in the Figure 2, By Theorem 1 , $S=\{u_1,u_3,u_5\}$ is a minimum irredundant set. Also, by Theorem 2.2, $ir(M(P_6)=|S|=\lceil\frac{6}{2}\rceil=3$.

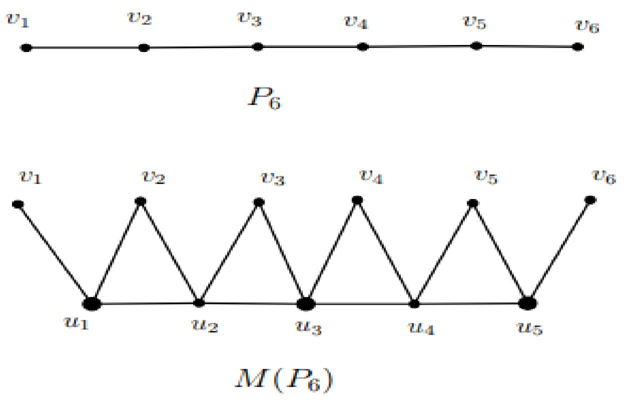

Fig. 2

Theorem 3: Let $vv_1v_2\cdots v_n$ be the star $K_{1,n}$ where, $v_1,v_2,\cdots,v_n$ are the end vertices and v be the full vertex in $K_{1,n}$, and $u_i=vv_i,\le i\le n$. Then in $M(K_{1,n}), S=\{u_1,u_2,\cdots,u_n\}$ is a minimum irredundant set for $n\ge 2$.

Proof. Given, star $K_{1,n}$ is $vv_1v_2\cdots v_n$ where, $v_1,v_2,\cdots,v_n$ are the end vertices and v be the full vertex in $K_{1,n}$, and $u_i=vv_i,\le i\le n$. Then $V(M(K_{1,n}))=\{v,v_i,u_j/1\le i\le n,1\le j\le n\}$. In $M(K_{1,n})$ each u_i is adjacent to v and v_i for $1\le i\le n$ and u_i is adjacent to u_j where $i\neq j$ and $1\le j\le n$ in

$M(K_{1,n})$. Hence, $G[u_1, u_2, \cdots, u_n]$ is a complete graph in $M(K_{1,n})$. Clearly, $|V(M(K_{1,n}))| = 2n+1$ and $|E(M(K_{1,n}))| = \frac{n(n+3)}{2}$.

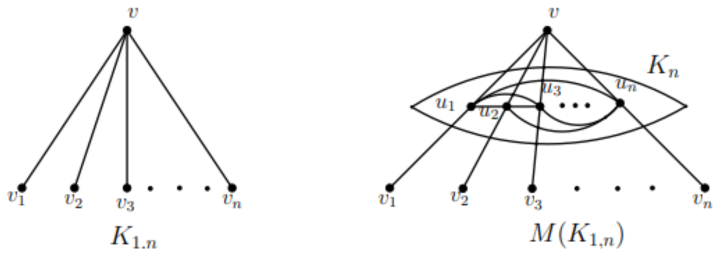

Fig. 3

Here, $N[u_1] = \{v, v_1, u_1, u_2, \cdots, u_n\}, N[u_2] = \{v, v_2, u_1, u_2, \cdots, u_n\}, N[u_3] = \{v, v_3, u_1, u_2, \cdots, u_n\}$. Proceeding like this we get $N[u_i] = v, v_i, u_1, u_2, \cdots, u_n$ where $1 \le i \le n$. Clearly, $N[u_1] \ne N[u_2] \ne N[u_3] \ne \cdots \ne N[u_n]$. Hence, we obtain a set $S = \{u_1, u_2, \cdots, u_n\}$ where $N[S] = V(M(K_{1,n}))$. Clearly, $pn[u_1, S] = \{v_1\}, pn[u_2, S] = \{v_2\}, \cdots, pn[u_n, S] = \{v_n\}$. Clearly, for each vertex $u_i \in S$, $pn[u_i, S] \ne \phi$. Therefore, S is an irredundant set. Now, we show that S is a minimum irredundant set for $M(K_{1,n})$ that is to show that S is a maximal irredundant set with a minimum cardinality. Suppose, S is not a maximal irredundant set. Then there exists a vertex $w \in V - S$, where w can be either v or v_i, for which $S' = S \cup \{w\}$ is irredundant, $1 \le i \le n$. In particular that $pn[w, S'] \ne \phi$, for all $w \in S'$. If $w = v$, then $pn[w, S'] = pn[v, S'] = N[v] - N[S' - \{v\}] = \{u_1, u_2, \cdots, u_n\} - V(M(K_{1,n})) = \phi$. Similarly, if $w = v_i, 1 \le i \le n$, then $pn[w, S'] = pn[v_i, S'] = N[v_i] - n[S'\{v_i\}] = \{u_i\} - V(M(K_{1,n})) = \phi$. Hence, S' is a redundant set which is contradiction to S' is an irredundant set. Thus, S is a maximal irredundant set with a minimum cardinality. Hence, $S = \{u_1, u_2, \cdots, u_n\}$ is a minimum irredundant set for $M(K_{1,n})$ for $n \ge 2$.

Theorem 4: For the graph $M(K_{1,n}), (n \ge 2)$, $ir(M(K_{1,n})) = n$.

Proof. By Theorem 3, the minimum irredundant set for $M(K_{1,n})$ is $S = \{u_1, u_2, \cdots, u_n\}$. By the definition of irredundant number, $ir(M(K_{1,n})) = |S| = n$.

Example : Consider the graph $M(K_{1,6})$ given in the Figure 4, By Theorem 3, $S = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ is the minimum irredundant set. Also, by Theorem 4, $ir(M(K_{1,6})) = |S| = 6$.

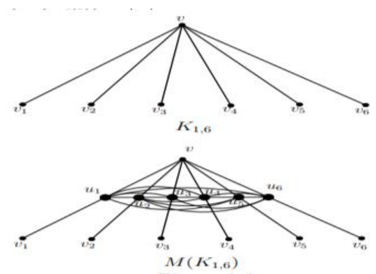

Fig. 4

Theorem 5: Let $v_1 v_2 \cdots v_n v_1$ be the cycle C_n and $u_i = v_i v_{i+1}, 1 \le i \le n$. Then, in $M(C_n)$, for $n \ge 3$, $S = \{u_i, u_{i+2}, u_{i+4}, \cdots, u_{i+\{\lceil \frac{n}{2} \rceil - 1\}2}\}$ is the minimum irredundant set for $1 \le i \le n$ and the suffices modulo n.

Proof. Given, $v_1v_2\cdots v_nv_1$ be the cycle C_n and $u_i = v_iv_{i+1}, 1 \le i \le n$, where the suffices modulo n. Then $V(M(C_n)) = \{v_i, u_i / 1 \le i \le n\}$. In $M(C_n)$, u_1 is adjacent to u_2, u_n, v_1 and v_2; u_i is adjacent to u_{i-1}, u_{i+1}, v_i and v_{i+1} for $2 \le i \le n-1$, u_n is adjacent to u_1, u_{n-1}, v_n and v_1, v_i is adjacent to u_i and u_{i-1} for $1 \le i \le n-1$ and v_n is adjacent to u_n and u_{n-1}, where $(u_i) = 4$ and $(v_i) = 2$ in $M(C_n)$. Thus, $u_1, u_2, \cdots, u_n$ induces a cycle of length n. Clearly, $|V(M(C_n))| = 2n$ and $|E(M(C_n))| = 3n$.

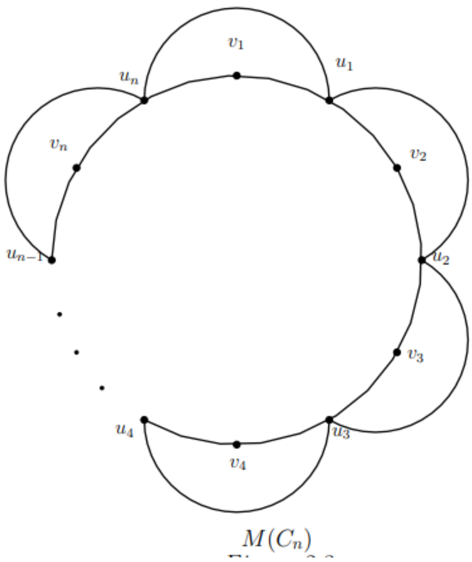

Fig. 5

By the construction of $M(C_n)$, starting with the vertex u_i for $1 \le i \le n$, we have $N[u_i] = \{v_i, v_{i+1}, u_i, u_{i-1}, u_{i+1}\}$ and $|N[u_i]| = 5$. Hence, the next vertex to be chosen as u_{i+2}, where $N[u_{i+2}] = \{v_{i+2}, v_{i+3}, u_{i+2}, u_{i+1}, u_{i+3}\}$. Clearly, $N[u_i] \ne N[u_{i+2}]$. Proceeding like this, we can choose the set as $S = \{u_i, u_{i+2}, u_{i+4}, \cdots, u_{i+\{\lceil \frac{n}{2} \rceil - 1\}2}\}$ where $N[S] = V(M(C_n))$. Here, $pn[u_i, S] = \{v_i, v_{i+1}\} \ne \phi, pn[u_{i+2}, S] = \{v_{i+2}, v_{i+3}\} \ne \phi$ and so on. Thus, it is clear that for every $u \in S, pn[u, S] \ne \phi$. Therefore, S is an irredundant set. Now, we show that S is a maximal irredundant set. Suppose, S is not a maximal irredundant set. Then, there exists a vertex $w \in V - S$, where w can be either v_i or u_j, for which $S' = S \cup \{w\}$ is irredundant, $1 \le i \le n$ and $u_j \in \{u_i / 1 \le i \le n\} - S$. In particular that $pn[w, S'] \ne \phi$ for all $w \in S'$. If $w = v_i, 1 \le i \le n$, then $pn[w, S'] = pn[v_i, S'] = N[v_i] - N[S' - \{v_i\}] = \{u_{i-1}, u_i\} - V(M(C_n)) = \phi$. Similarly, if $w = u_j, u_j \in \{u_i / 1 \le i \le n\} - S$, then $pn[w, S'] = \phi$. Hence, S' is a redundant set which is contradiction to S' is an irredundant set. Hence, S is a redundant set which is contradiction to S is an irredundant set. Thus, S is a maximal irredundant set with a minimum cardinality. Hence, $S = \{u_i, u_{i+2}, u_{i+4}, \cdots, u_{i+\{\lceil \frac{n}{2} \rceil - 1\}2}\}$ is the minimum irredundant set for $1 \le i \le n$ and the suffices modulo n. Hence the proof.

Theorem 6: For the graph $M(C_n), (n \ge 3)$, $ir(M(C_n) = \lceil \frac{n}{2} \rceil$.

Proof. By Theorem 3, the minimum irredundant set for $M(C_n)$ is $S = \{u_i, u_{i+2}, u_{i+4}, \cdots, u_{i+\{\lceil \frac{n}{2} \rceil - 1\}2}\}$ where $1 \le i \le n$ and the suffices modulo n. By the definition of irredundant number, $ir(M(C_n) = |S| = \lceil \frac{n}{2} \rceil$.

Example : Consider the graph $M(C_6)$ given in the Figure 6, By Theorem 3 $S=\{u_1,u_3,u_5\}$ is the minimum irredundant set. Also, by Theorem 4, $ir(M(C_6)=|S|=\lceil \frac{6}{2} \rceil = 3$.

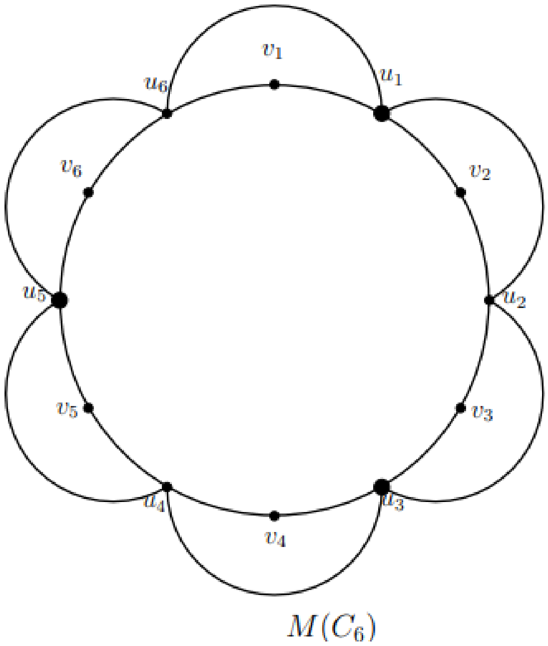

Fig. 6

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