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A Modern Approach To Fractional-Order Kinetic Equations With Hyper-Bessel Function Solutions And Application On Environmental Field

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Abstract
This study presents a novel approach to solving fractional-order kinetic equations (FKEs) by employing Hyper-Bessel functions in conjunction with the Sumudu transform. The proposed method leverages the Sumudu transform to obtain analytical solutions of FKEs, wherein the function f(t) is expressed in terms of a Hyper-Bessel function. The resulting solutions are general in nature and can be applied to a wide range of existing as well as newly formulated fractional kinetic equations, with potential applications in environmental science and related fields.Special functions included in fractional kinetic equations have been shown to be helpful in the explanation and resolution of numerous important mathematical and mathematical physics issues. Because arbitrary-order kinetic equations are so important, the goal of this study is to use the Sumudu Transform technique to solve a new fractional-order kinetic equation involving the Hyper Bessel function with their fractional derivatives. MATLAB-generated graphical representations used in our analysis to demonstrate the behavior of these solutions under various parametric values. The outcomes of the study are highly flexible and may lead to both confirmed and maybe undiscovered research findings in this field.
Key words: *Hyper-Bessel function, Fractional order derivative, fractional differential equations , Kinetic equation MAT Lab software .*

1. INTRODUCTION

Fractional kinetic equations (FKEs) are mathematical models used to describe reaction rates and transport processes. These equations often involve derivatives of non-integer order, enabling a more accurate representation of complex natural phenomena compared to classical integer-order models. Fractional calculus provides a mathematical framework for addressing fractional differential equations, which involve derivatives and integrals of non-integer order. These equations play a crucial role in applied sciences, particularly in dynamic systems, control theory, mathematical physics, and engineering. In recent years, they have been extensively employed to develop mathematical models for various physical phenomena. However, the nonlocal nature of fractional derivatives makes solving these equations challenging. Consequently, special functions and numerical methods are often utilized to obtain approximate solutions. Researchers continue to develop innovative methods and tools to enhance the understanding and solution of such problems. Over the past few decades, fractional-order differential equations have demonstrated significant applicability across numerous branches of engineering and physical sciences. Environmental science employs a wide range of applications, including scientific research, data acquisition, and technological innovations, to address complex environmental challenges. These applications encompass understanding ecological processes and developing advanced technologies for pollution mitigation and sustainable resource management

Deliberate the problem of modelling pollutant dispersion in a river system. A fractional-order model offers a more accurate representation of complex transport and diffusion processes compared to classical integer-order models. By employing a Hyper-Bessel function to characterize the pollutant diffusion rate and applying the Sumudu transform, an analytical solution can be derived to describe the temporal and spatial variation of pollutant concentration. Such a solution provides valuable insights for predicting the pollutant's impact on the river ecosystem and developing effective mitigation strategies.

Fractional kinetic equations (FKEs) have numerous significant applications in modeling complex physical systems and are widely employed in engineering, management, physical sciences, and social sciences. Over the years, extensive research has been conducted on FKEs involving various special functions, resulting in substantial contributions to the literature of fractional calculus. Notably, the works of Haubold and Mathai [9] and Saxena and Kalla [15] marked important advancements in this domain. Several scholars have rigorously investigated FKEs to uncover new applications and theoretical developments. Among them are Haubold and Mathai [1], Baricz and Mehrez [3–4], Ozarslan [6], Gupta and Parihar [7], Gupta et al. [8], Kumar et al. [11], Saxena et al [12] Nisar et al. [13], Saichev and Zaslavsky [14], Saxena et al.[16], Chand et al. [22], Samraiz et al. [17] Sharma and Bhargava [18–20], Suthar et al. [21], Srivastava and Tomovski [10], Meena and Purohit [25], Jarad and Abdeljawad investigated a modified Laplace transform for certain generalized fractional operators. Their research has focused on the extension of FKEs associated with special functions, leading to several important and intriguing findings. Mathematical equations that govern the time evolution of a system while incorporating fractional derivatives to capture non-local effects and long-range interactions. Fractional calculus and fractional kinetic equations (FKEs) are increasingly applied in environmental science to model complex phenomena such as pollutant dispersion in water bodies, groundwater flow through porous media, soil contamination, and anomalous diffusion in ecosystems. Fundamentally, fractional-order kinetics offers an advanced framework for analyzing and modeling a broad spectrum of environmental processes, facilitating more precise predictions and enhanced management of environmental resources.

Saxena and Kalla [15] defined generalized FKE as

$$N(t) - N_0 f(t) = -c^{v}\, {}_0D_t^{-v}\, N(t), \mathrm{Re}(v) > 0 \tag{1}$$

$N(t)$: Number density of species at the time t, $N_0 = N(t=0), c \neq 0$ is a constant, $f(t) \in L(0,\infty)$ and ${}_0D_t^{-v}$ is the fractional differential operator [12].

Due to the importance of Hyper-Bessel functions which appeared in the various field of applied and pure mathematical frameworks. Hyper-Bessel has several uses in science and engineering and is frequently used to solve fractional order differential equations. Here, in this paper, we consider the function f(t) as a Hyper-Bessel function and acquire the solution of FKE with utilizing the powerful Sumudu Transform [24] technique. In mathematical modeling, kinetic equations serve as fundamental tools in mathematical physics and the natural sciences, describing the continuity of motion of materials. In this study, new solutions of a generalized Hadamard fractional kinetic equation involving the generalized k -Bessel function are presented using the Mellin transform technique.

The Hyper-Bessel function [5] in generalized form defined as

$$\mathfrak{J}^{(\vartheta)}_{\tau_1,\tau_2,\dots,\tau_\vartheta}(\xi) = \left(\frac{\xi}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i} \widetilde{\mathfrak{J}}^{(\vartheta)}_{\tau_1,\tau_2,\dots,\tau_\vartheta}(\xi)$$

where

$$\widetilde{\mathfrak{J}}^{(\vartheta)}_{\tau_1,\tau_2,\dots,\tau_\vartheta}(\xi) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\Gamma(\ell+\tau_1+1)\dots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{\xi}{\vartheta+1}\right)^{\ell(\vartheta+1)} \tag{2}$$

where $\xi, \tau_i \in \mathbb{C}, \Re(\tau_i+1) > 0, |\xi| < \infty, i = 1,2,\dots,\vartheta$. For more details, we refer [3, 4, 5].

The fractional derivative [12] of order λ of the function $f(t) = t^{\beta}$ is given by

$$D^{\lambda}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\lambda+1)}t^{\beta-\lambda} \quad ; \Re(\beta) > -1, 0 < \Re(\lambda) < 1, t > 0 \tag{3}$$

So, in view of (2) and (3) we have

$${}_0D_t^{\lambda}\left(\mathfrak{J}^{(\vartheta)}_{\tau_1,\tau_2,\dots,\tau_\vartheta}(\xi)\right) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\Gamma(\ell+\tau_1+1)\dots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)}$$

$$\times \frac{\Gamma\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)+1\right)}{\Gamma\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)-\lambda+1\right)} (\xi)^{\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)-\lambda} \quad (4)$$

The Sumudu Transform [24] is described as follows over the set 'Ω' of functions as

$$S[f(t)] = G(u) = \frac{1}{u}\int_0^{\infty} f(t)e^{-t/u}\,dt \qquad ;\ 0<t<\infty, u\in(-\tau_1,\tau_2)$$

(5)

where $\Omega = \{f(t)|\ Me^{|t|/\tau_j}\ ; if\ t\in(-1)k\times[0,\infty)\}, M\ is\ a\ constant\ and\ \tau_1,\tau_2>0$

We need the following to support our major findings:

Lemma 1:

$$S\left(\Im_{\tau_1,\tau_2,\dots,\tau_\vartheta}^{(\vartheta)}(\xi)\right) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)}{\Gamma(\ell+\tau_1+1)\dots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)} (u)^{\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)} \quad (6)$$

Lemma 2:

$$S\left[{}_0D_t^{\lambda}\left(\Im_{\tau_1,\tau_2,\dots,\tau_\vartheta}^{(\vartheta)}(\xi)\right)\right] = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)}{\Gamma(\ell+\tau_1+1)\dots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)} (u)^{\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)-\lambda\right)} \quad (7)$$

Proof: In view of definition (2), (4), (5) and doing simple calculations we can obtain (6) and (7) easily.

In this manuscript, we acquire the findings in terms of ML function [23], defined as

$$E_{\theta,\delta}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(l\theta+\delta)}\ ;\ \Re(\theta),\Re(\delta)>0, \theta,\delta\ \epsilon\ \mathbb{C} \quad (8)$$

2. Main Results

Theorem1: If $c>0, v>0, |t|<\infty, \Re(\tau_i+1)>0, \Re(\tau)>0, \Re(\zeta)>0, t,\tau_i,\zeta,\eta,\xi\in\mathbb{C}, r\in\mathbb{R}, \eta\in(0,1)\cup\mathbb{N}$, $\Re(\xi)>0$, then the FKE

$$N(t) - N_0\left\{\Im_{\tau_1,\tau_2,\dots,\tau_\vartheta}^{(\vartheta)}(t)\right\} = -c^{v}\,{}_0D_t^{-v}\,N(t) \quad (9)$$

and its solution is

$$N(t) = N_0\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)t^{\ell(\vartheta+1)+\sum_{i=1}^{\vartheta}\tau_i}}{\Gamma(\ell+\tau_1+1)\dots\Gamma(\ell+\tau_\vartheta+1)\ell!\,(\vartheta+1)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)}} \times E_{v,\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1}(-c^v t^v) \quad (10)$$

Proof: Using (5) on (9),

$$N(u) = N_0\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)}{\Gamma(\ell+\tau_1+1)\dots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{u}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)} (1+c^v u^v)^{-1}$$

$$= N_0\sum_{j=0}^{\infty} \frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)}{\Gamma(\ell+\tau_1+1)\dots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{u}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)} \sum_{\alpha=0}^{\infty}(-c^v u^v)^{\alpha}$$

$$= N_0\sum_{j=0}^{\infty} \frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)}{\Gamma(\ell+\tau_1+1)\dots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)} \sum_{\alpha=0}^{\infty}(-c^v)^{\alpha} u^{\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+v\alpha\right)}$$

(11)

Now using the definition of inverse Sumudu Transform and with simple evaluation we achieve the result (10).

3.1 Mathematical Analysis and Explanation

We obtain several values of N(t) for (10), by varying t while keeping v constant. These values are interpreted in table 1 and the 3D and 2D graphs 1(a), 1(b), which illustrate the behavior of the result for the kinetic equation (9).

Table 1: "The values of $N(t)$ with fix v for (10)"

t	"$N(t)$ at $v = 0.1$"	"$N(t)$ at $v = 0.5$"	"$N(t)$ at $v = 0.9$"	"$N(t)$ at $v = 1.3$"
0	0	0	0	0
0.08	2.40324E+15	-2.4822E+14	2.68905E+13	-2.91384E+12
0.16	3.63689E+15	-4.85009E+14	6.89933E+13	-9.85605E+12
0.24	4.74184E+15	-7.30177E+14	1.21619E+14	-2.04129E+13
0.32	5.78971E+15	-9.8714E+14	1.83711E+14	-3.45584E+13
0.4	6.91295E+15	-1.25741E+15	2.54859E+14	-5.23604E+13
0.48	7.84798E+15	-1.54188E+15	3.34905E+14	-7.39265E+13
0.56	8.89422E+15	-1.85112E+15	4.23812E+14	-9.93848E+13
0.64	9.95756E+15	-2.15559E+15	5.21612E+14	-1.28875E+14
0.72	1.1042E+16	-2.49563E+15	6.28378E+14	-1.62545E+14
0.8	1.21504E+16	-2.83156E+15	7.44211E+14	-2.00556E+14
0.88	1.32849E+16	-3.19364E+15	8.69228E+14	-2.43033E+14
0.96	1.44471E+16	-3.57213E+15	1.00356E+15	-2.90163E+14
1.04	1.56384E+16	-3.96726E+15	1.14736E+15	-3.42094E+14
1.12	1.68597E+16	-4.37925E+15	1.30076E+15	-3.98988E+14
1.2	1.81121E+16	-4.80835E+15	1.46393E+15	-4.61005E+14
1.28	1.93963E+16	-5.25475E+15	1.63702E+15	-5.28307E+14
1.36	2.0713E+16	-5.71868E+15	1.82019E+15	-6.01057E+14
1.44	2.20629E+16	-6.20036E+15	2.01362E+15	-6.79418E+14
1.52	2.34464E+16	-6.7E+15	2.21748E+15	-7.63555E+14
1.6	2.48642E+16	-7.21782E+15	2.43194E+15	-8.53632E+14
1.68	2.63167E+16	-7.75405E+15	2.65716E+15	-9.49816E+14
1.76	2.78045E+16	-8.30889E+15	2.89335E+15	-1.05227E+15
1.84	2.93279E+16	-8.88258E+15	3.14066E+15	-1.16117E+15
1.92	3.08875E+16	-9.47535E+15	3.39929E+15	-1.27667E+15
2	3.24837E+16	-1.00874E+16	3.66943E+15	-1.39896E+15

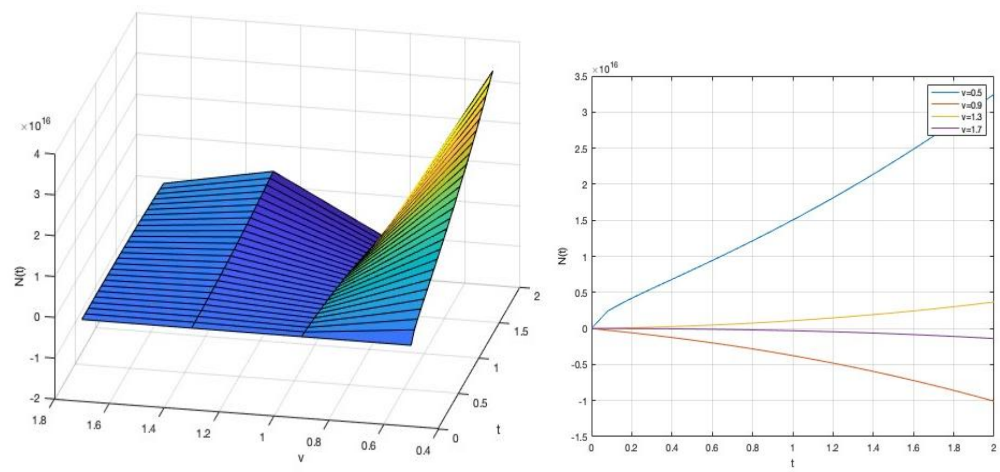


Figure 1 **(a) 3D graph for (10)** **(b) 2D graph for (10)**

Theorem-2: If $c > 0, v > 0, |t| < \infty, \Re(\tau_i + 1) > 0, \Re(\tau) > 0, \Re(\xi) > 0$ $t, \tau_i, \zeta, \eta, \xi \in \mathbb{C}, d \neq c, d > 0, r \in \mathbb{R}, \eta \in (0,1) \cup \mathbb{N}, \Re(\zeta) > 0$, then the FKE

$$N(t) - N_0 \left\{ \mathfrak{I}^{(\vartheta)}_{\tau_1, \tau_2, \ldots, \tau_\vartheta} (d^v t^v) \right\} = -c^v \, {}_0D_t^{-v} \, N(t)$$

(12)

and its solution is

$$N(t) = N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \Gamma\left(v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)\right)+1\right)}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_{\vartheta}+1)\ell!} \times \left(\frac{dt}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta} \tau_i+\ell(\vartheta+1)\right)} E_{v,\left(v\left(\sum_{i=1}^{\vartheta} \tau_i+\ell(\vartheta+1)\right)+1\right)}(-c^v t^v) \tag{13}$$

Proof: Using (5) on (12),

$$N(u) = N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \Gamma\left(v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)\right)+1\right)}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_{\vartheta}+1)\ell!} \left(\frac{ud}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta} \tau_i+\ell(\vartheta+1)\right)} (1+c^v u^v)^{-1}$$

$$= N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \Gamma\left(v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)\right)+1\right)}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_{\vartheta}+1)\ell!} \left(\frac{d}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta} \tau_i+\ell(n+1)\right)} \sum_{\alpha=0}^{\infty} (-c^v u^v)^{\alpha} u^{\left(v\left(\sum_{i=1}^{\vartheta} \tau_i+\ell(\vartheta+1)\right)\right)}$$

$$= N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \Gamma\left(v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)\right)+1\right)}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_{\vartheta}+1)\ell!} \left(\frac{d}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta} \tau_i+\ell(\vartheta+1)\right)} \sum_{\alpha=0}^{\infty} (-c^v)^{\alpha} u^{\left(v\left(\sum_{i=1}^{\vartheta} \tau_i+\ell(\vartheta+1)\right)+v\alpha\right)}$$

Now using the definition of inverse Sumudu Transform, (8) and then with simple evaluation we achieve the result (13).

3.2 Mathematical Analysis and Explanation

We obtain several values of N(t) for (13), by varying t while keeping v constant. These values are interpreted in table 2 and the 3D and 2D graphs 2(a), 2(b), which illustrate the behavior of the result for the kinetic equation (12).

Table 2: "The values of $N(t)$ with fix v for equation (13)"

t	"$N(t)$ at $v = 0.1$"	"$N(t)$ at $v = 0.5$"	"$N(t)$ at $v = 0.9$"	"$N(t)$ at $v = 1.3$"
2	0.001114394	-0.001479266	0.001694018	-0.002810911
2.04	0.001411325	-0.001889534	0.002204592	-0.003683596
2.08	0.001779217	-0.002402634	0.002854421	-0.004803091
2.12	0.002233185	-0.003041721	0.003677696	-0.006232666
2.16	0.002791193	-0.003834604	0.004716125	-0.008050212
2.2	0.003474527	-0.004814576	0.006020399	-0.010351277
2.24	0.004308341	-0.006021378	0.007651925	-0.013252652
2.28	0.005322272	-0.007502306	0.009684839	-0.016896616
2.32	0.006551151	-0.009313494	0.012208367	-0.021455935
2.36	0.008035809	-0.011521387	0.015329575	-0.027139747
2.4	0.009823997	-0.014204434	0.01917656	-0.03420046
2.44	0.011971426	-0.017455028	0.023902162	-0.042941822
2.48	0.014542956	-0.021381732	0.029688251	-0.053728359
2.52	0.017613937	-0.026111817	0.036750692	-0.066996362
2.56	0.021271729	-0.031794168	0.045345067	-0.083266678
2.6	0.025617424	-0.038602589	0.055773274	-0.103159561
2.64	0.030767788	-0.046739569	0.068391107	-0.127411884
2.68	0.036957457	-0.056440575	0.083616974	-0.156898063
2.72	0.044041411	-0.067978914	0.101941881	-0.192648072
2.76	0.052497766	-0.081671262	0.123940883	-0.235883997
2.8	0.062430915	-0.097883936	0.150286173	-0.28804063
2.84	0.074075067	-0.117039992	0.181762044	-0.350805664
2.88	0.087698228	-0.139627267	0.219281966	-0.426159128
2.92	0.103606678	-0.166207477	0.263908058	-0.516419801
2.96	0.122150002	-0.1974265	0.316873269	-0.624298407
3	0.143726743	-0.234025996	0.379606636	-0.752958527

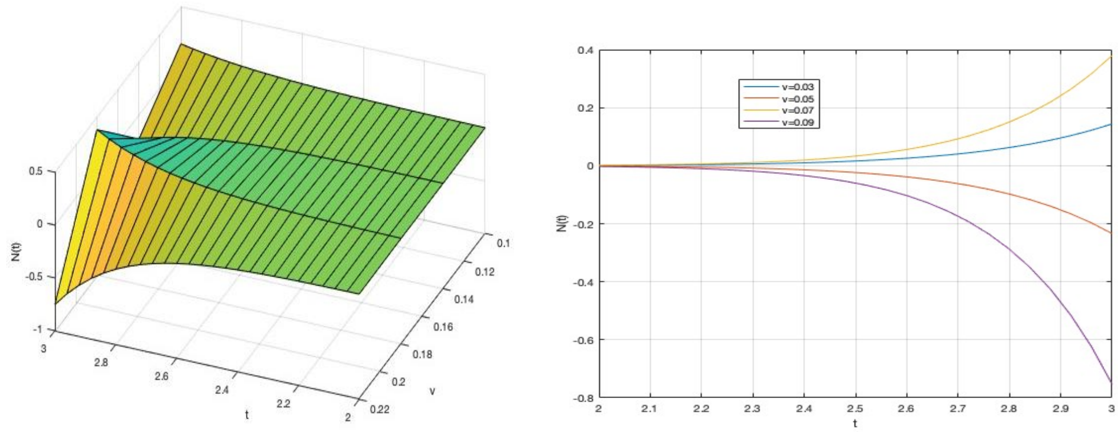


Figure 2 (a) 3D graph for equation (13) (b) 2D graph for equation (13)

Theorem-3: If $c > 0, v > 0, |t| < \infty, \Re(\tau_i + 1) > 0, \Re(\tau) > 0, \Re(\zeta) > 0, t, \tau_i, \zeta, \eta, \xi \in \mathbb{C}, r \in \mathbb{R}, \eta \in (0,1) \cup \mathbb{N}, \ \Re(\xi) > 0, \lambda \neq v$, then the FKE

$$N(t) - N_0 \left({}_0D_t^{\lambda} \left(\mathfrak{J}^{(\vartheta)}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}(t) \right) \right) = -c^v \, {}_0D_t^{-v} \, N(t) \tag{14}$$

and its solution is

$$N(t) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) + 1\right) t^{\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) - \lambda}}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_\vartheta+1)\ell!} \left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)} E_{v, \sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) - \lambda + 1}(-c^v t^v) \tag{15}$$

Proof: Using (5) on (14),

$$N(s) = N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) + 1\right)}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_\vartheta+1)\ell!} \left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)} (u)^{\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) - \lambda\right)} \times (1 + c^v u^v)^{-1}$$

$$= N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) + 1\right)}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_\vartheta+1)\ell!} \left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)} (u)^{\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) - \lambda\right)} \sum_{\alpha=0}^{\infty} (-c^v u^v)^\alpha$$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) + 1\right)}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_\vartheta+1)\ell!} \left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)} \sum_{k=0}^{\infty} (-c^v)^\alpha (u)^{\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) - \lambda + v\alpha\right)}$$

Now taking inverse Sumudu Transform, we have

$$N(t) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) + 1\right) t^{\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) - \lambda}}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_\vartheta+1)\ell!} \left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)} \times \sum_{\alpha=0}^{\infty} \frac{(-c^v t^v)^\alpha}{\Gamma\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) - \lambda + 1 + v\alpha\right)}$$

By using (8), we get the result (15).

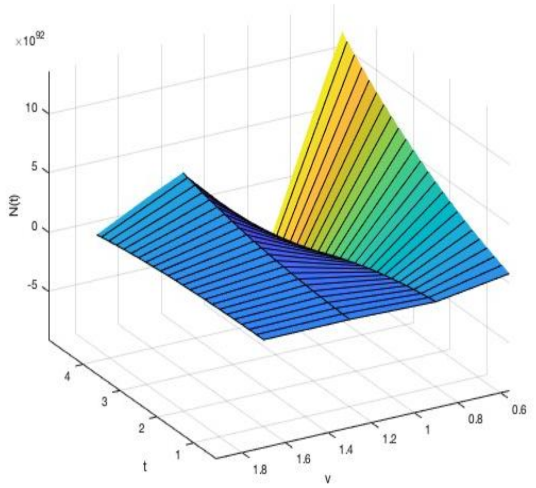
3.3 Mathematical Analysis and Explanation

Several values of N(t) corresponding to equation (15) are computed by varying t while keeping $\boldsymbol{v}$ constant. These results are presented in Table 3, and their behavior for the kinetic equation (14) is illustrated through the 2D and 3D plots shown in Figures 3(a) and 3(b).

Table 3: "The values of $N(t)$ with fix v for (15)"

t	"$N(t)$ at $v = 0.1$"	"$N(t)$ at $v = 0.5$"	"$N(t)$ at $v = 0.9$"	"$N(t)$ at $v = 1.3$"
0	0	0	0	0
0.2	5.88804E+91	-7.86187E+90	1.12174E+90	-1.61069E+89

0.4	9.38709E+91	-1.60481E+91	2.99548E+90	-5.66357E+89
0.6	1.27784E+92	-2.51300E+91	5.47438E+90	-1.21451E+90
0.8	1.62491E+92	-3.52106E+91	8.54498E+90	-2.12180E+90
1	1.98662E+92	-4.63432E+91	1.22152E+91	-3.30807E+90
1.2	2.36626E+92	-5.85667E+91	1.65018E+91	-4.78446E+90
1.4	2.76576E+92	-7.19136E+91	2.14209E+91	-6.60273E+90
1.6	3.18643E+92	-8.66145E+91	2.69966E+91	-8.75493E+90
1.8	3.62926E+92	-1.02099E+92	3.32506E+91	-1.12734E+91
2	4.09504E+92	-1.18996E+92	4.02061E+91	-1.41806E+91
2.2	4.58449E+92	-1.37136E+92	4.78874E+91	-1.74992E+91
2.4	5.09825E+92	-1.5655E+92	5.63191E+91	-2.12523E+91
2.6	5.63695E+92	-1.77269E+92	6.55269E+91	-2.54630E+91
2.8	6.20120E+92	-1.99325E+92	7.55367E+91	-3.01548E+91
3	6.79162E+92	-2.22752E+92	8.63755E+91	-3.53517E+91
3.2	7.40882E+92	-2.47584E+92	9.80706E+91	-4.10778E+91
3.4	8.05343E+92	-2.73857E+92	1.10650E+92	-4.73578E+91
3.6	8.72610E+92	-3.01609E+92	1.24143E+92	-5.42169E+91
3.8	9.42750E+92	-3.30878E+92	1.38580E+92	-6.16806E+91
4	1.01583E+93	-3.61704E+92	1.53991E+92	-6.97751E+91
4.2	1.09193E+93	-3.94129E+92	1.70407E+92	-7.85273E+91
4.4	1.17111E+93	-4.28194E+92	1.87861E+92	-8.79644E+91
4.6	1.25345E+93	-4.63946E+92	2.06386E+92	-9.81547E+91
4.8	1.33904E+93	-5.01430E+92	2.26018E+92	-1.09007E+92
5	1.42795E+93	-5.40694E+92	2.46792E+92	-1.20671E+92



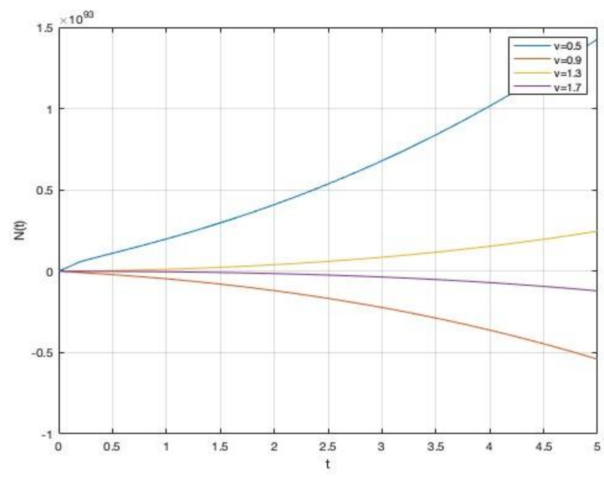


Figure 3 **(a) 3D graph for equation (15)** **(b) 2D graph for equation (15)**

Theorem-4: If $c > 0, d > 0, v > 0, |t| < \infty, \Re(\tau_i + 1) > 0, \Re(\tau) > 0, \Re(\zeta) > 0, t, \tau_i, \zeta, \eta, \xi \in \mathbb{C}, \eta \in (0,1) \cup \mathbb{N}, \Re(\xi) > 0, \lambda \neq v$, $d \neq c$, then the FKE

$$N(t) - N_0 \left({}_0D_t^{\lambda} \left(\mathfrak{J}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}^{(\vartheta)} (d^v t^v) \right) \right) = -c^v {}_0D_t^{-v} N(t) \tag{16}$$

is given as

$$N(t) = N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \Gamma\left(v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)\right)+1\right)}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_{\vartheta}+1)\ell!} \left(\frac{d}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta} \tau_i+\ell(\vartheta+1)\right)} \times (t)^{v\left(\sum_{i=1}^{\vartheta} \tau_i+\ell(\vartheta+1)\right)-\lambda} E_{v,v\left(\sum_{i=1}^{\vartheta} \tau_i+\ell(\vartheta+1)\right)-\lambda+1}(-c^v t^v) \tag{17}$$

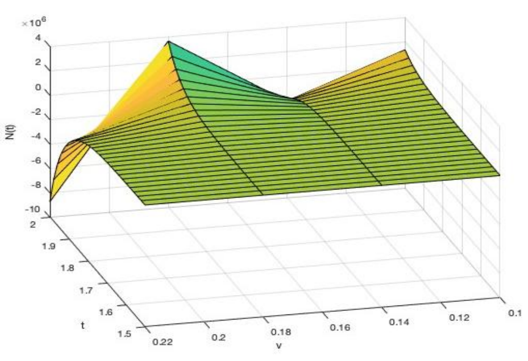
Proof: Doing the same process as we have done for Theorem 1 and using Lemma 2, we can achieve the result equation (17).

3.4 Mathematical Analysis and Explanation

We obtain several values of N(t) for (17), by varying t while keeping v constant. These values are interpreted in table 4 and the 3D and 2D graphs 4(a), 4(b), which illustrate the behavior of the result for the kinetic equation (16).

Table 4: "The values of $N(t)$ with fix v for equation (17)"

t	"$N(t)$ at $v = 0.1$"	"$N(t)$ at $v = 0.5$"	"$N(t)$ at $v = 0.9$"	"$N(t)$ at $v = 1.3$"
1.5	989.2757986	-1373.487548	1953.719148	-4275.797537
1.52	1361.174549	-1900.522248	2732.823474	-6004.230874
1.54	1865.720527	-2619.686998	3807.102295	-8397.381892
1.56	2547.886046	-3597.621875	5282.937249	-11698.71367
1.58	3467.196015	-4923.032803	7303.288022	-16236.82745
1.6	4702.257497	-6713.712035	10059.75851	-22454.02663
1.62	6356.663494	-9125.729139	13808.51895	-30944.1534
1.64	8566.685676	-12365.45611	18841.29636	-42502.65047
1.66	11511.29846	-16705.29561	25763.02816	-58152.73967
1.68	15425.24578	-22504.26281	35028.28162	-79432.85474
1.7	20616.08617	-30234.85096	47489.21556	-108112.1204
1.72	27486.45034	-40518.36982	64208.75967	-146742.875
1.74	36563.14558	-54171.02501	86593.8889	-198662.1817
1.76	48565.27807	-72264.56394	116505.4846	-268298.2248
1.78	64304.28938	-96205.90739	156403.448	-361522.809
1.8	85049.78541	-127842.0256	209538.6739	-486118.3581
1.82	112316.3733	-169598.3546	280207.486	-652397.5468
1.84	148128.5501	-224661.9443	374089.5801	-874026.9443
1.86	195143.199	-297224.4811	498697.9799	-1169124.134
1.88	256852.7113	-392805.7617	663979.7551	-1561722.584
1.9	337856.561	-518685.7008	883120.4064	-2083732.689
1.92	444225.8423	-684483.3711	1173624.437	-2777574.627
1.94	583994.647	-902936.0961	1558771.96	-3699724.181
1.96	767825.3192	-1190951.947	2069589.389	-4925504.086
1.98	1009913.218	-1571037.608	2747525.97	-6555581.284
2	1329224.026	-2073444.023	3648103.645	-8724810.316



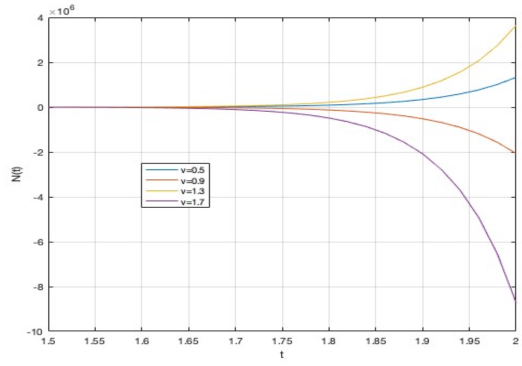


Figure 4. (a) 3D graph for equation (17) (b) 2D graph for equation (17)

4. Specific Cases

(i) Substituting $d = c$ in (12), then the FKE reduces as

$$N(t) - N_0\left\{\mathfrak{I}^{(\vartheta)}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}(c^v t^v)\right\} = -c^v\,{}_0D_t^{-v}\,N(t) \tag{18}$$

and its solution is

$$N(t) = N_0\sum_{\ell=0}^{\infty}\frac{(-1)^\ell\Gamma\left(v\left(\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1)\right)+1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\times\left(\frac{ct}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)} \times E_{v,\left(v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)+1\right)}(-c^v t^v) \tag{19}$$

(ii) Substituting $d = c$ in (16), then the FKE reduces in

$$N(t) - N_0\left({}_0D_t^{\lambda}\left(\mathfrak{I}^{(\vartheta)}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}(c^v t^v)\right)\right) = -c^v\,{}_0D_t^{-v}\,N(t) \tag{20}$$

and its solution is

$$N(t) = N_0\sum_{\ell=0}^{\infty}\frac{(-1)^\ell\Gamma\left(v\left(\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1)\right)+1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{c}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)} \times (t)^{v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)-\lambda}\times E_{v,v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)-\lambda+1}(-c^v t^v) \tag{21}$$

Additional special cases of our results can be obtained by substituting suitable parameter values into the corresponding special function; however, these cases are not presented here explicitly. Fig 5 show that insight into temporal dynamics of system by depicting how the reaction rates change over the time.

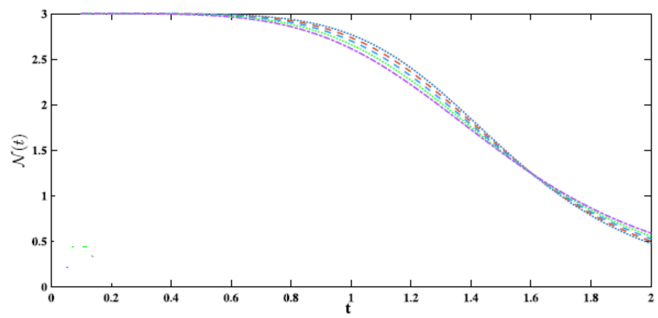


Figure 5. Plot diagram N (t) and t

5. CONCLUSION

In this work, we propose using the Sumudu transform technique to solve a class of new generalized fractional kinetic equations that procedure the Hyper-Bessel function and its fractional derivatives. The mittag-Leffler function is used to direct the final solutions. This method is very important since fractional kinetic equations are widely applicable in many different scientific and engineering fields. After being expressed in terms of Hyper-Bessel functions, the solution can be examined and applied to particular environmental issues. This new method to fractional-order kinetic equations may be useful in the environmental field since it uses Hyper-Bessel functions as solutions. This method makes use of techniques like the Sumudu transform, of which the Hyper-Bessel function is an essential component, to solve these equations. These solutions can be used to mimic environmental phenomena, such as diffusion and transport processes, which are often described by kinetic equations. Along with finding these solutions, we use MATLAB to provide a thorough inspection of their behavior through numerical and graphical representations under various parametric conditions. By generalizing the idea of integer-order calculus, fractional calculus offers a more profound framework for comprehending a range of real-world occurrences and basic scientific concepts. Due to its comprehensive application in a variety of fields, such as control systems, elasticity, electric drives, circuit theory, continuum mechanics, heat transfer, quantum mechanics, fluid dynamics, signal processing, biomathematics, biomedical engineering, social systems, and bioengineering, fractional calculus research has attracted a lot of consideration recently.

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