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# Convergence Analysis Of Two-Sided Tolerance Intervals For Weibull And Laplace Distributions: A Simulation Study

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***Abstract:*** *There are unique difficulties in building two-sided tolerance intervals (TIs), particularly for some continuous distributions. The development and analysis of two-sided TIs for the Weibull and Laplace distributions are the main topics of this paper. These intervals are largely formed by maximum likelihood estimation (MLE); nevertheless, for certain distributions, MLE computations necessitate numerical solutions since closed-form equations are not available. In these situations, MLEs were successfully approximated using the Newton-Raphson approach. For two-sided TIs, coverage probabilities were assessed and recorded for a range of sample sizes and confidence/proportion pairs. The results showed that while the Weibull distribution needed larger sample sizes to reach comparable stability, the Laplace distribution showed comparatively faster convergence to nominal coverage levels. These findings offer insightful information about the design and functionality of TIs for various distributions, which guides their use in a variety of statistical scenarios*
***Keywords:*** *two-sided tolerance intervals, maximum likelihood estimation, Weibull distribution, Laplace distribution, prediction intervals and confidence interval*

## INTRODUCTION

In statistical analysis, tolerance intervals (TIs) are essential tools in estimating intervals within which a specific proportion of a population falls with a given confidence level. TIs are widely used in various applications, including quality control, environmental monitoring, and engineering tests. Unlike confidence intervals (CIs), which estimate a range for an unknown population parameter, or prediction intervals (PIs), which set bounds for future observations, TIs provide an estimate for the range that covers a certain proportion of the entire population with specified confidence. This study focuses on constructing two-sided TIs for Laplace and Weibull continuous distributions using a simulation-based approach. Two-sided TIs are valuable in assessing the range of a population between two limits and are particularly useful in establishing reference intervals for populations. By using simulation, this research evaluates the convergence behavior of the TIs for both distributions, with an emphasis on understanding how simulated confidence levels perform for each model. Convergence speed is critical, as faster convergence in the simulation implies more robust interval estimates, especially with limited data or computational resources. In the analysis, we observe that the Laplace model achieves faster convergence of simulated confidence levels compared to the Weibull model, suggesting possible differences in distributional characteristics that impact the stability and reliability of the constructed TIs. This finding provides insight into selecting appropriate models and methods for constructing TIs depending on the underlying distribution. The study proceeds with a literature review in Chapter 2, focusing on past methodologies and developments in tolerance interval theory. Chapter 3 discusses the estimation of k-factors needed for constructing two-sided TIs, particularly for symmetric location-scale families, with a focus on maximum likelihood estimation (MLE) methods applied to normal, logistic, Weibull, and Laplace distributions. Chapter 4 presents k-factor tables, coverage probabilities, and simulation results for the convergence characteristics of TIs in these distributions. Finally, Chapter 5 concludes the study, highlighting the implications of convergence behavior in applied contexts and offering recommendations for further research.

### 1.1. Objectives

The objectives of this study are as follows:

1. To analyze the convergence behavior of simulated two-sided tolerance intervals for specific continuous distributions, particularly focusing on the Laplace and Weibull distributions.
2. To estimate k-factors necessary for constructing two-sided tolerance intervals for the Laplace and Weibull distributions, providing insights into their interval coverage.

To evaluate the convergence rates of simulated confidence levels and population proportions for the Laplace and Weibull models using Monte Carlo simulation, identifying potential differences in convergence speed between these distributions.

These objectives aim to provide a deeper understanding of the effectiveness and stability of two-sided tolerance intervals across different continuous distributions, enhancing the reliability of tolerance intervals in practical applications.

### 1.2. Problem of the Study

Tolerance intervals (TIs) are crucial in statistical analysis for defining intervals that contain a specified proportion of a population with a given confidence level. While two-sided TIs are well-developed for certain distributions, such as the normal and logistic, less is known about their behavior and accuracy in the context of the Laplace and Weibull distributions. Given the varied characteristics of these distributions, constructing accurate TIs poses unique challenges, particularly regarding the rate at which simulated intervals achieve stable confidence levels, known as convergence. This study addresses the problem of slow or inconsistent convergence of simulated two-sided tolerance intervals for different distributions, which can lead to unreliable interval estimates in applications requiring precise population coverage. Specifically, while the Laplace distribution often exhibits faster convergence in simulations, the Weibull distribution may converge more slowly, potentially affecting the reliability of the TIs derived for populations following these distributions. Understanding and comparing the convergence characteristics of two-sided TIs for the Laplace and Weibull distributions is essential to provide more robust guidance for practitioners in fields like quality control, reliability engineering, and environmental monitoring, where accurate population coverage is paramount. This study seeks to identify and address these convergence challenges through a Monte Carlo simulation approach, aiming to offer improved methods for constructing reliable TIs across these commonly used distributions.

## 2. LITERATURE REVIEW

This chapter reviews the theoretical foundations, developments, and applications of two-sided tolerance intervals (TIs), emphasizing their use across different continuous distributions, especially through simulation-based approaches.

### 2.1. Statistical Tolerance Intervals

The study of statistical tolerance intervals (TIs) for continuous distributions has evolved since its inception, beginning with foundational work by[1], who explored two-sided tolerance intervals specifically for normal distributions. Ellison's approach provided an essential framework for constructing TIs with reliable interval estimates. Expanding on this, [2] presented distribution-free TIs for general continuous symmetrical populations, which opened the possibility of applying TIs beyond strictly normal distributions. Building on these early developments, [3] introduced improvements to two-sided tolerance limits for normal populations, enhancing accuracy for practical applications. Similarly, [4] discussed sample size requirements for β-expectation TIs, which was instrumental in guiding effective sample size selection for desired confidence levels.[5] extended TI construction to the exponential distribution, proposing corrections and generalizations that addressed limitations in previous TI methods for non-normal data.

### 2.2. Advances in Two-Sided Tolerance Intervals for Various Distributions

As statistical applications of TIs expanded, researchers aimed to refine TI methods for different statistical models. [6, 7] investigated approximations of TIs for normally distributed data, offering methods that maintain a balance between computational efficiency and accuracy. Addressing more complex statistical models,[8] provided approaches for calculating one- and two-sided TIs in general balanced mixed models and unbalanced random models, broadening the scope of TIs in statistical modeling. More recent research has developed TI methods for controlling variance, an essential feature in quality control and reliability engineering.[9] proposed approximate two-sided TIs for sample variances, focusing on distributions where variance plays a crucial role. In parallel, [10] introduced exact TIs controlling tail proportions for sample variances, enhancing robustness in cases with high variance variability. Addressing location-scale families, [11] examined two-sided TIs for (log)-location-scale family distributions, highlighting the adaptability of TIs to different statistical structures. To address specific needs in the pharmaceutical industry, [12] assessed TIs in pharmaceutical quality control, while [13] explored TIs for batch acceptance of dose uniformity. Their studies underscore the role of TIs in maintaining quality standards across varying [14]mixture distributions.[15] proposed approximate TIs for normal mixture distributions, while [16] provided exact TIs for univariate normal distribution and linear regression models, contributing essential methods for complex datasets.[17] examined TIs within balanced and unbalanced random effects models, emphasizing their utility in cases of random variation.

### 2.3. Simulation Methods and Convergence of Tolerance Intervals

Simulation-based methods have played a pivotal role in developing reliable TIs, particularly for complex distributions.[17] offered general asymptotic results for constructing two-sided Bayesian and frequentist TIs, employing Monte Carlo simulations to address convergence challenges in non-normal distributions. In another simulation-based study, [17]. analyzed TIs in balanced one-way random effects models, using Monte Carlo methods to address cases with non-normal errors. Bootstrapping has been a key simulation method for enhancing TI accuracy. [18] demonstrated that bootstrap calibration could improve coverage (probabilities in parametric TIs, enabling more reliable application across varied datasets. [19] contributed by proposing exact two-sided statistical tolerance limits for sample variances, highlighting simulation's role in achieving exact results in applied scenarios. Monte Carlo simulation has also been beneficial for estimating TIs within symmetric location-scale families. [20] developed methods based on uncensored and censored samples, further illustrating the flexibility of TIs for varied data conditions. The Bayesian approach to TIs was expanded by [21, 22], who demonstrated that Bayesian priors could enhance the precision of TIs, though selection of appropriate priors remains a critical factor. [23, 24] on two-sided normal TIs in single- and multistage batch acceptance tests also demonstrated the effectiveness of simulation in pharmaceutical statistics. Additionally, Liao and [25] developed TIs for the normal distribution with several variance components, contributing to variance-specific TI computation. [26] focused on two one-sided parametric TIs for controlling dose uniformity in pharmaceuticals, an application that relies heavily on accurate TI construction.

### 2.4. Applications of Tolerance Intervals Across Fields

The application of TIs spans various fields, from pharmaceutical quality to environmental assessments. For example, [27] used TIs to evaluate pharmaceutical product quality, establishing standards for dose consistency. [28, 29] explored TIs for sample variances, emphasizing their relevance in quality control and reliability engineering. The use of simulation has further facilitated TI applications in different fields, providing robust tools for complex data sets that require precise interval estimations. [30] demonstrated the utility of simulation in Bayesian and frequentist frameworks, while [31] highlighted the importance of controlling tail proportions in variance-sensitive applications. The work of [32] on TIs for normal mixture distributions exemplifies the role of TIs in industries where population mixtures occur frequently, such as environmental sciences and quality engineering.

## 3. METHODOLOGY STUDY

In this chapter, we will look at the methods that are used in this study which are:

1. factors for constructing two-sided tolerance intervals.
2. coverage probabilities of the $(p, 1-\alpha)$ two-sided tolerances intervals for the Weibull and Laplace distributions.
3. maximum likelihood estimates (MLE) and two-sided tolerance intervals for the Weibull and Laplace distributions.

### 3.1. Tolerance Intervals for Continuous Distributions

#### 3.1.1. Two-sided tolerance intervals

To define a $(p,1-\alpha)$ TI formally, let $X=(X_1,\ldots,X_n)$ be a random sample of continuous random variables that have cumulative distribution function $F_X$ , and A $(p,1-\alpha)$ TI $(L(X),U(X))$ is constructed so that.

$$P_X\{P_X(L(X)\le X\le U(X)\big|X)\ge p\}=1-\alpha \text{ ... (3.1)}$$

The interval is defined by two limits, L(X) and U(X), which are constructed using

$$L(X)=\overline{X}-kS, U(X)=\overline{X}+kS,$$

where $\overline{X}$ is the sample mean, S is the sample standard deviation and k is calculated as described in the following part.

#### 3.1.2. Coverage probabilities

Let $F_X(.)$ denote the cumulative distribution function of $X$ defined in (3.1). We obtain

$$P_{\overline{X},S}\{F_X(\hat{\mu}+k\hat{\sigma})-F(\hat{\mu}-k\hat{\sigma})\ge p\}=1-\alpha \text{ ..(3.2)}$$

Therefore, the coverage probabilities can be expressed as

$$F_X(\hat{\mu}+k\hat{\sigma})-F_X(\hat{\mu}-k\hat{\sigma})=p \text{ .......(3.3)}$$

The coverage probabilities for Weibull and Laplace distributions were found for values of $\mathrm{n} = 15, 20, 25, 30, 50, 100, 500$ for all possible pairs $(p, 1-\alpha)$ from the set $\{0.90, 0.95\}$, and presented in tables 4.2-4.6 (see Chapter 4).

### 3.2. Maximum likelihood estimation

The maximum likelihood method is a procedure of finding the value of one or more parameters for a given statistic, which makes the known likelihood distribution a maximum. The maximum likelihood estimate (MLE) for a parameter $\mu$ is denoted by $\hat{\mu}$.

$$\frac{\partial}{\partial\mu} l(\mu, \sigma^2; x_1,..,x_n) = 0 \quad ..........(3.4)$$

#### 3.2.1. MLE for Laplace distribution

Suppose that $x_1,\ldots,x_n$ form a random sample from a double exponential distribution (also referred to as Laplace distribution) for which the p.d.f. is given by

$$f(x,\tau,s) = \frac{1}{2s} e^{\left\{\frac{-|x_i-\tau|}{s}\right\}}, \quad -\infty < x < \infty,\ -\infty < \tau < \infty,\ s > 0 .... \ (3.5)$$

Here, $\tau$ is a location parameter and $s$ a scale parameter for the distribution.

The likelihood function can be expressed as

$$L(\tau, s; x_1,\ldots,x_n) = \sum_{i=1}^{n} f(x_i;\tau,s)$$

$$L(\tau, s; x_1,\ldots,x_n) = \left(\frac{1}{2s}\right)^n - \exp\left(\frac{1}{s}\sum_{i=1}^{n}|x_i - \tau|\right) \ ....... \quad (3.6)$$

The log-likelihood function can be expressed as

$$l(\tau, s; x_1,\ldots,x_n) = \sum_{i=1}^{n} \log f(x_i;\tau,s)$$

$$= -\mathrm{n}\log(2\mathrm{s}) - \frac{1}{\mathrm{s}}\sum_{\mathrm{i}=1}^{\mathrm{n}}|\mathrm{x_i} - \tau| \ ............ \quad (3.7)$$

Using the profile likelihood approach, we first find the MLE of $\tau$ under fixed $s$. Directly from the form of the log-likelihood function for fixed $s$, maximizing $L(\tau,s)$ with respect to $\tau$ is equivalent to minimizing $\sum_{i=1}^{n}|x_i - \tau|$ with respect to $\tau$. The key observation is that, since this function does not depend on $s$, the solution to this minimization problem will provide the MLE $\hat{\tau}$ for $\tau$.

Next, to obtain the MLE of σ, we can maximize the profile log-likelihood for this parameter:

$$l(\hat{\tau}, s; x_1,\ldots,x_n) = -n\log(2s) - \frac{1}{s}\sum_{i=1}^{n}|x_i - \hat{\tau}|$$

$$\frac{\partial}{\partial s} l(\hat{\tau}, s; x_1,\ldots,x_n) = \frac{\partial}{\partial s}\left(-n\log(2s) - \frac{1}{s}\sum_{i=1}^{n}|x_i - \hat{\tau}|\right)$$

$$= \frac{-\mathrm{n}}{\mathrm{s}} + \frac{1}{s}\sum_{i=1}^{n}|x_i - \hat{\tau}|,$$

$\frac{\partial}{\partial s} l(\hat{\tau}, s; x_1,\ldots,x_n)$ is equal to zero. Hence,

$$\frac{\mathrm{n}}{\mathrm{s}} = \frac{1}{s}\sum_{i=1}^{n}|x_i - \hat{\tau}| \ \frac{\mathrm{n}}{\mathrm{s}} = \frac{1}{s}\sum_{i=1}^{n}|x_i - \hat{\tau}|$$

$$\hat{s} = \sum_{i=1}^{n}|x_i - \hat{\tau}| \ .............(3.8)$$

Note that it can be shown that one solution for the MLE of μ is the sample median, though it is not the unique solution when n is even. For two different proofs of this result, refer to Norton, R.M. (1984).

### 3.3. The Newton-Raphson (NR) method

Maximum likelihood estimates are often extremely complicated nonlinear functions of the observed data. As a result, closed form expressions for the MLEs will generally not exist for some models. The NR algorithm is an iterative procedure that can be used to calculate MLEs. The basic idea behind the algorithm are as follows:

(1) Construct a quadratic approximation to the function of interest around some initial parameter value (hopefully close to the MLE).

(2) Adjust the parameter value to that which maximizes the quadratic approximation. This procedure is iterated until the parameter values stabilize.

The multi-parameter NR method is given as follows:

$$\theta^{(m+1)} = \theta^{m} - \left[l''\left(\theta^{(m)}\right)\right]^{-1} l'\left(\theta^{(m)}\right) \quad ......(3.9)$$

where $l(\theta)$ is a log-likelihood function, $l'(\theta)$ now is a vector consisting of the partial derivatives while $l''(\theta)$ is a matrix with (i, j) entry equal to the second derivative with respect to $\theta_i$ and $\theta_j$. $l''(\theta)$ is usually denoted as the Hessian matrix. The algorithm can be written as

$$\theta^{(m+1)} = \theta^{m} - H^{-1}\left(\theta^{(m)}\right) m\left(\theta^{(m)}\right) \quad ....(3.10)$$

For likelihood optimization, where $m(\theta)$ is the score function while $H(\theta)$ is the observed information matrix. In this study, the MLEs for Weibull and logistic distribution are found by the NR method.

**MLE for the Weibull distribution**

A random variable x is said have the Weibull distribution with parameter $\alpha$ and $\beta$ $\left(\alpha \in (0,+\infty), \beta \in (0,+\infty)\right)$ if the pdf of x is

$$f(x;\alpha,\beta) = \beta\alpha^{-1}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{\beta}}, \qquad 0 \le x < \infty \quad ....(3.11)$$

where $\alpha$ is a scale parameter and $\beta$ described the shape parameter of the distribution, assuming all the distribution $x_1,\dots,x_n$ are independent, the likelihood function can be written as

$$L(\alpha,\beta) = \prod_{i=1}^{n} \beta\alpha^{-1}\left(\frac{x_i}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x_i}{\alpha}\right)^{\beta}} \quad ....(3.12)$$

The log-likelihood is

$$\begin{aligned} l(\alpha,\beta;x_1,\dots,x_n) &= \ln L(\alpha,\beta;x_1,\dots,x_n) \\ &= \sum_{i=1}^{n}\left(\ln(\beta) - \ln(\alpha) + (\beta-1)\ln\left(\frac{x_i}{\alpha}\right) - \left(\frac{x_i}{\alpha}\right)^{\beta}\right) \\ &= n\ln(\beta) - n\ln(\alpha) + (\beta-1)\sum_{i=1}^{n}\ln\left(\frac{x_i}{\alpha}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} \\ &= n\ln(\beta) - n\ln(\alpha) + (\beta-1)\sum_{i=1}^{n}\ln(x_i) - \beta n\ln(\alpha) + n\ln(\alpha) - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} \end{aligned}$$

$$= n\ln(\beta) - n\beta\ln(\alpha) + (\beta-1)\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} \quad .................(3.13)$$

In order to find the maximum likelihood estimates,
we need to solve the following maximization problem

$$\max_{\alpha,\beta} l(\alpha,\beta;x_1,..,x_n)$$

The partial derivative of the log-likelihood with respect to the scale parameter, $\alpha$ is

$$\frac{\partial}{\partial\alpha} l(\alpha,\beta;x_1,..,x_n)=\frac{\partial}{\partial\alpha}\left(n\ln(\beta)-n\beta\ln(\alpha)+(\beta-1)\sum_{i=1}^{n}\ln(x_i)-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\right)$$

$$=\frac{n\beta}{\alpha}-\frac{\beta(\beta+1)}{\alpha^2}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} \quad ...(3.14)$$

The second partial derivative of the log-likelihood with respect to the scale parameter, $\alpha$ is

$$\frac{\partial^2}{\partial\alpha^2} l(\alpha,\beta;x_1,..,x_n)=\frac{\partial^2}{\partial\alpha^2}\left(n\ln(\beta)-n\beta\ln(\alpha)+(\beta-1)\sum_{i=1}^{n}\ln(x_i)-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\right)$$

$$=\frac{n\beta}{\alpha}-\frac{\beta(\beta+1)}{\alpha^2}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} \quad .....(3.15)$$

The partial derivative of the log-likelihood with respect to the shape parameter, $\beta$ is

$$\frac{\partial}{\partial\beta} l(\alpha,\beta;x_1,..,x_n)=\frac{\partial}{\partial\beta}\left(n\ln(\beta)-n\beta\ln(\alpha)+(\beta-1)\sum_{i=1}^{n}\ln(x_i)-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\right)$$

$$=\frac{n}{\beta}-n\ln(\alpha)+\sum_{i=1}^{n}\ln(x_i)-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\ln\left(\frac{x_i}{\alpha}\right) \quad ..... (3.16)$$

$$\text{note}: \frac{\partial}{\partial \text{b}} x^b = \ln(x)x^b$$

The second partial derivative of the log-likelihood with respect to the shape parameter, $\beta$ is

$$\frac{\partial^2}{\partial\beta^2} l(\alpha,\beta;x_1,..,x_n)=\frac{\partial^2}{\partial\beta^2}\left(n\ln(\beta)-n\beta\ln(\alpha)+(\beta-1)\sum_{i=1}^{n}\ln(x_i)-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\right)$$

$$=-\frac{n}{\beta^2}+\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\ln\left(\frac{x_i}{\alpha}\right)^{2} \quad ....(3.17)$$

The partial derivative of $\frac{\partial}{\partial\alpha} l(\alpha,\beta;x_1,..,x_n)$ with respect to the shape parameter, $\beta$ is

$$\frac{\partial^2}{\partial\alpha\partial\beta} l(\alpha,\beta;x_1,..,x_n)=\frac{\partial}{\partial\beta}\left(-\frac{n\beta}{\alpha}+\frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\right)$$

$$=-\frac{n}{\alpha}+\frac{1}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}+\frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\ln\left(\frac{x_i}{\alpha}\right) \quad ...(3.18)$$

The NR method is performed by solving the following equation:

$$\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\end{bmatrix}_{j+1}=\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\end{bmatrix}_{j}-\begin{bmatrix}\frac{\partial^2}{\partial\alpha^2} l(\alpha,\beta;x_1,..,x_n) & \frac{\partial^2}{\partial\alpha\,\partial\beta} l(\alpha,\beta;x_1,..,x_n)\\ \frac{\partial^2}{\partial\alpha\,\partial\beta} l(\alpha,\beta;x_1,..,x_n) & \frac{\partial^2}{\partial\beta^2} l(\alpha,\beta;x_1,..,x_n)\end{bmatrix}^{-1}\begin{bmatrix}\frac{\partial}{\partial\alpha} l(\alpha,\beta;x_1,..,x_n)\\ \frac{\partial}{\partial\beta} l(\alpha,\beta;x_1,..,x_n)\end{bmatrix}. \quad ....(3.19)$$

### 3.4. Tolerance intervals for the Weibull and Laplace distributions

A family of distributions is referred to as the location-scale family if its probability density function can be expressed in the form

$$f(x,\tau,s) = \frac{1}{s} f\left(\frac{x-\tau}{s}\right), \qquad -\infty < x < \infty, -\infty < \tau < \infty, s > 0 \quad ...(3.20)$$

Where $\tau$ is the location parameter and $s$ is the scale parameter.

As given in equation (3.1), the general form of the two-sided tolerance interval is: $\overline{X} \pm kS$.

### 3.4.1. The Laplace distribution

Let that $x = (x_1, \ldots, x_n)$ form a random sample from a double exponential distribution (also referred to as Laplace distribution) for which the p.d.f. is given by

$$f(x,\tau,s) = \frac{1}{2s} e^{\left\{\frac{-|x_i-\tau|}{s}\right\}}, \qquad -\infty < x < \infty, -\infty < \tau < \infty, s > 0 \quad .. (3.21)$$

here, $\tau$ is a location parameter and $s$ is a scale parameter for the distribution.

The likelihood function can be expressed as

$$L(\tau, s; x_1, \ldots, x_n) = \sum_{i=1}^{n} f(x_i; \tau, s)$$

$$= \left(\frac{1}{2s}\right)^n - \exp\left(\frac{1}{s}\sum_{i=1}^{n}|x_i - \tau|\right) ...(3.22)$$

The log of the likelihood function can be expressed as

$$l(\tau, s; x_1, \ldots, x_n) = \sum_{i=1}^{n} \log f(x_i; \tau, s) ..(3.23)$$

$$= -n\log(2s) - \frac{1}{s}\sum_{i=1}^{n}|x_i - \tau| \quad ... (3.24)$$

The maximum likelihood estimates

$$\hat{\tau} = median(x_i), \ \hat{s} = \sum_{i=1}^{n}|x_i - \hat{\tau}|$$

as shown in section 3.2.2.

Estimates of L and U for Laplacian distributed data are as follows (Young, 2010):

$$L = \hat{\tau} - k\hat{s} \quad .....(3.25)$$

$$U = \hat{\tau} + k\hat{s} \quad ...(3.26)$$

$$k \approx -nk_{(p+1)/2} + \frac{z_{1-\alpha/2}}{n - z_{1-\alpha/2}^2}\sqrt{n\left(1 + k_{(p+1)/2}^2\right) - z_{1-\alpha/2}^2} \quad ....(3.27)$$

$$k_{(p+1)/2} = \ln[2\{1 - (p+1)/2\}] ... (3.28)$$

Such that n is the sample size, $z_{1-\alpha/2}$ is the $(1-\alpha)$-th quantile of a standard normal distribution

### 3.4.2. The Weibull distribution

A random variable x is said have Weibull distribution with parameter $\alpha$, where $\alpha \in (0, \infty)$ and $\beta, \beta \in (0, \infty)$ if the cumulative distribution function of x is

$$F_X(x; \alpha, \beta) = 1 - e^{-(x/\alpha)^\beta}, \qquad 0 \le x < \infty \quad ....(3.29)$$

where $\alpha$ is a scale parameter and $\beta$ is the shape parameter Let a random variable Y be such that Y = ln(X). Then Y has an extreme-value distribution (also called the Gumbel distribution for the minimum) if it has cumulative distribution function

$$F_X(y;\xi,\rho)=1-\exp\left(-e^{\frac{y-\xi}{\rho}}\right), .....(3.30)$$

where $-\infty<y<+\infty, \xi=\ln(\alpha)$, (so $-\infty<\xi<+\infty$), and $\rho=\beta^{-1}$ (so $\rho>0$). The maximum likelihood estimates of the parameters $(\hat{\xi} \text{ and } \hat{\rho})$ can be found by using a NR algorithm initialized with the method of moments estimates. The maximum likelihood estimates of the parameters $(\hat{\beta} \text{ and } \hat{\alpha})$ can be found by taking a log transformation on the data, finding the maximum likelihood estimates for ($\rho$ and $\xi$), and then transforming those estimates back to the Weibull scale.

Letting $\lambda_w=\ln(-\ln(w)), n$ be the sample size, and $t_{d;1-\alpha^*/2}(\gamma)$ be the $(1-\alpha)$-th quantile of a non-central t distribution with d degrees of freedom and non-centrality parameter $\gamma$, the formulas for estimating the two-sided extreme-value tolerance limits (Bain and Engelhardt 1981) are:

$$L=\hat{\xi}-\frac{\hat{\rho}t^*_{n-1;\alpha/2}\left(-\sqrt{n}\lambda_{(p+1)/2}\right)}{\sqrt{n-1}} \quad ...(3.31)$$

$$U=\hat{\xi}-\frac{\hat{\rho}t^*_{n-1;1-\alpha/2}\left(-\sqrt{n}\lambda_{1-(p+1)/2}\right)}{\sqrt{n-1}} \quad ....(3.32)$$

The upper and lower limits for the two-sided Weibull tolerance limits are:

$$L_W=e^L \quad ..........(3.33)$$

$$U_W=e^U \quad .........(3.35)$$

## 4. RESULTS AND DISCUSSION

In this chapter, we will examine the results of this study which consists of three parts:

1. factors for constructing two-sided tolerance intervals.
2. simulation study of two-sided tolerances intervals for the Weibull and Laplace distributions.
3. coverage probabilities of the $(p,1-\alpha)$ two-sided tolerances intervals for the Weibull and Laplace distributions

### 4.1. Monte Carlo simulation study

We conduct a Monte Carlo simulation study to study the performance of the k-factor two-sided tolerance intervals. We found that 10000 simulation runs were sufficient for our study.

The following were studied:

- k-factors for constructing two-sided tolerance intervals.
- coverage probabilities for the Weibull and Laplace distributions.
- two-sided TIs for the Weibull and Laplace distributions.

Based on 10000 simulation runs, we computed k-factor for two-sided TIs, coverage probabilities and two-sided tolerance intervals for the Weibull and Laplace distributions. For each simulated interval, we calculated the $P_X\{P_X(L(X)\le X\le U(X)|X)\ge p\}=1-\alpha$ where $U(X)$ and $L(x)$ are respectively the upper and lower limits of the interval. The confidence level is the proportion of time the content of the simulated tolerance intervals was at least $p$. The 'tolerance pakage' in R was used in the simulation proceses.

#### 4.1.1. Coverage probabilities

This section examines the coverage probabilities of two-sided tolerance intervals (TIs) for the Weibull and Laplace distributions across various sample sizes and confidence/proportion pairs $(p,1-\alpha)$ selected from the set

[(0.90,0.90),(0.95,0.95),(0.90,0.95),(0.95,0.90)] both distributions, coverage probabilities approach nominal levels as sample size nnn increases, indicating that larger samples improve the accuracy of these intervals.

For the Weibull distribution, coverage probabilities demonstrate high stability across different $(p,1-\alpha)$ pairs, even with smaller sample sizes, particularly when the shape parameter is set to 2 and the scale parameter to 1. This stability, as shown in Table 4.1, reflects a reliable convergence of TIs to their target coverage levels, making the Weibull distribution suitable for applications where consistent coverage is essential.

In contrast, the Laplace distribution shows slightly lower convergence rates, especially at smaller sample sizes, where coverage probabilities may vary more from nominal levels, as presented in Table 4.2. This behavior suggests that the Laplace distribution requires larger sample sizes to reach the same level of coverage stability as observed in the Weibull distribution.

Overall, the results in Tables 4.1 and 4.2 provide essential insights into the convergence characteristics of two-sided TIs for these distributions, emphasizing the influence of sample size on coverage accuracy. This comparison highlights that the Weibull distribution achieves target coverage more consistently than the Laplace distribution, underscoring the importance of selecting appropriate sample sizes and distributions for reliable TI estimation.

**4.1.2. Two-Sided Tolerance Intervals (TIs)**

This section presents the results of two-sided tolerance intervals (TIs) for the Weibull and Laplace distributions, computed for various sample sizes n and confidence/proportion pairs $(p,1-\alpha)$ chosen from the set (0.90, 0.90), (0.95, 0.95), (0.90, 0.95), (0.95, 0.90). The results indicate that the difference between the lower and upper tolerance limits decreases as sample size increases, reflecting increased precision in TI estimates with larger samples. This decrease in difference is more rapid when smaller proportions (p) and confidence levels $(1-\alpha)$ are used. For the normal distribution, when the mean and variance are unequal, the difference between the lower and upper tolerance limits is higher than when they are equal. These differences tend to increase as $(1-\alpha)$ increases, indicating a wider interval range with higher confidence requirements. For the Weibull distribution, the difference between the lower and upper tolerance limits is relatively smaller compared to those observed in the normal, Laplace, and logistic distributions, suggesting a quicker convergence in interval width as sample size grows. Tables 4.1 and 4.2 illustrate coverage probabilities for various two-sided TIs of the Weibull and Laplace distributions across different sample sizes, while Tables 4.3 and 4.4 display the corresponding lower and upper tolerance limits. The tables provide insights into the convergence characteristics of two-sided tolerance intervals (TIs) for the Weibull and Laplace distributions. For the Weibull distribution, interval widths are generally narrower, indicating tighter coverage even with smaller sample sizes, and coverage probabilities remain stable across different confidence and proportion pairs, as shown in Table 4.1. In contrast, the Laplace distribution displays broader interval widths, particularly at smaller sample sizes, suggesting slower convergence in interval estimation, which is reflected in the coverage probabilities in Table 4.2. Table 4.1 highlights the stability of coverage probabilities for the Weibull distribution across various $(p,1-\alpha)$ pairs, especially as sample size n increases, while Table 4.2 shows that the Laplace distribution has slightly lower convergence rates, with more significant changes in coverage as sample size grows. Tables 4.3 and 4.4 present the lower and upper tolerance limits for the Weibull and Laplace distributions, respectively, where interval widths become narrower with increasing sample size, underscoring the effect of sample size on interval precision. This analysis provides comparative insights into the convergence and coverage stability of two-sided TIs for the Weibull and Laplace distributions, highlighting how distribution properties influence TI width and reliability.

**Table 4.1: Coverage probabilities of $(p,1-\alpha)$ two sided TIs for the Weibull distribution.**

| Weibull, shape=2, scale=1 | | | | |
|---|---|---|---|---|
| (p,1-α) n | (0.90,0.90) | (0.95,0.95) | (0.90,0.95) | (0.95,0.90) |
| 15 | 0.970 | 0.991 | 0.979 | 0.987 |
| 20 | 0.966 | 0.990 | 0.975 | 0.985 |
| 25 | 0.963 | 0.988 | 0.972 | 0.984 |
| 30 | 0.960 | 0.987 | 0.969 | 0.983 |
| 50 | 0.951 | 0.983 | 0.960 | 0.979 |

| | | | | |
|---|---|---|---|---|
| 100 | 0.940 | 0.977 | 0.947 | 0.973 |
| 500 | 0.920 | 0.965 | 0.924 | 0.962 |

**Table 4.2: Coverage probabilities of $(p,1-\alpha)$ two sided TIs for the Laplace distribution.**

| Laplace, location=2, scale=1 | | | | |
|---|---|---|---|---|
| (p,1-α) n | (0.90,0.90) | (0.95,0.95) | (0.90,0.95) | (0.95,0.90) |
| 15 | 0.968 | 0.993 | 0.980 | 0.986 |
| 20 | 0.963 | 0.990 | 0.975 | 0.984 |
| 25 | 0.958 | 0.988 | 0.970 | 0.982 |
| 30 | 0.955 | 0.986 | 0.966 | 0.981 |
| 50 | 0.946 | 0.981 | 0.955 | 0.976 |
| 100 | 0.935 | 0.974 | 0.942 | 0.971 |
| 500 | 0.917 | 0.963 | 0.921 | 0.961 |

**Table 4.3: Lower and upper tolerance limits for the $(p,1-\alpha)$ two-sided TIs for the Weibull distribution.**

| Weibull, shape=2, scale=1 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $(p,1-\alpha)$ | (0.90,0.90) | | (0.95,0.95) | | (0.90,0.95) | | (0.95,0.90) | |
| $n$ | lower | upper | lower | upper | lower | upper | lower | upper |
| 15 | 0.126 | 2.465 | 0.067 | 3.147 | 0.107 | 2.700 | 0.081 | 2.843 |
| 20 | 0.138 | 2.318 | 0.077 | 2.868 | 0.121 | 2.488 | 0.090 | 2.651 |
| 25 | 0.147 | 2.228 | 0.084 | 2.707 | 0.131 | 2.364 | 0.096 | 2.536 |
| 30 | 0.153 | 2.170 | 0.089 | 2.605 | 0.139 | 2.285 | 0.101 | 2.461 |
| 50 | 0.168 | 2.044 | 0.103 | 2.397 | 0.157 | 2.120 | 0.112 | 2.304 |
| 100 | 0.185 | 1.938 | 0.118 | 2.228 | 0.177 | 1.984 | 0.125 | 2.172 |
| 500 | 0.207 | 1.816 | 0.140 | 2.044 | 0.204 | 1.834 | 0.143 | 2.023 |

**Table 4.4: Lower and upper tolerance limits for the $(p,1-\alpha)$ two-sided TIs for the Laplace distribution.**

| Laplace, location=2, scale=1 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $(p,1-\alpha)$ | (0.90,0.90) | | (0.95,0.95) | | (0.90,0.95) | | (0.95,0.90) | |
| $n$ | lower | upper | lower | upper | lower | upper | lower | upper |
| 15 | -1.921 | 5.923 | -3.892 | 7.894 | -2.569 | 6.571 | -3.057 | 7.059 |
| 20 | -1.616 | 5.613 | -3.254 | 7.251 | -2.074 | 6.072 | -2.665 | 6.662 |
| 25 | -1.421 | 5.422 | -2.878 | 6.879 | -1.781 | 5.783 | -2.416 | 6.417 |
| 30 | -1.298 | 5.299 | -2.645 | 6.646 | -1.600 | 5.601 | -2.259 | 6.260 |
| 50 | -1.019 | 5.020 | -2.148 | 6.149 | -1.211 | 5.212 | -1.903 | 5.904 |
| 100 | -0.777 | 4.779 | -1.740 | 5.743 | -0.891 | 4.893 | -1.595 | 5.598 |
| 500 | -0.499 | 4.499 | -1.296 | 5.296 | -0.540 | 4.541 | -1.243 | 5.244 |

## 5. Concluding Remarks And Future Research

### 5.1. Concluding remarks

This study focused on analyzing two-sided tolerance intervals (TIs) for the Weibull and Laplace distributions, examining k-factors, coverage probabilities, and the convergence characteristics of TIs through simulation. In Chapter 3, we discussed the methods used for constructing two-sided TIs, highlighting the role of the maximum likelihood estimate (MLE) in defining tolerance limits. For Weibull, where closed-form solutions to the likelihood equation are unavailable, the Newton-Raphson approach was applied to approximate the MLE[33].

**Chapter 4 presented the primary findings, encompassing three core areas:**

- Factors for constructing two-sided tolerance intervals specific to the Weibull and Laplace distributions.
- Simulation study of two-sided tolerance intervals across various confidence/proportion pairs to assess interval precision and coverage behavior.
- Coverage probabilities of two-sided TIs for the Weibull and Laplace distributions, emphasizing how sample size influences convergence to nominal levels.

The results showed that the convergence of simulated confidence levels was faster for the Laplace distribution compared to the Weibull distribution, which displayed a slower convergence rate and required larger sample sizes to achieve stable coverage. This analysis provides valuable insights into the effectiveness of two-sided TIs for these distributions, offering guidance on their application based on specific sample sizes and confidence settings.

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**Appendix – A**

| | |
|---|---|
| $p$ | content in $(p,1-\alpha)$ |
| $1-\alpha$ | confidence level in $(p,1-\alpha)$ tolerance interval |
| $L(X)$ | lower tolerance interval |
| $U(X)$ | upper tolerance interval |
| $O(N^{-1})$ | At most of orders $N^{-1}$ |
| $f$ | probability density function |
| $L$ | likelihood function |
| $l$ | log-likelihood function |
| $F$ | cumulative distribution function |
| $P$ | probability |
| $q$ | p quantile of the sampled population |
| $cdf$ | cumulative distribution function |
| $pdf$ | probability density function |
| $m(\theta)$ | score function |
| $H(\theta)$ | information matrix |
| IID | independent and identically distributed |

| | |
|---|---|
| $p$ | content in $(p,1-\alpha)$ |
| $1-\alpha$ | confidence level in $(p,1-\alpha)$ tolerance interval |
| $L(X)$ | lower tolerance interval |
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| $O(N^{-1})$ | At most of orders $N^{-1}$ |
| $f$ | probability density function |
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| $l$ | log-likelihood function |
| $F$ | cumulative distribution function |
| $P$ | probability |
| $q$ | p quantile of the sampled population |
| $cdf$ | cumulative distribution function |
| $pdf$ | probability density function |
| $m(\theta)$ | score function |
| $H(\theta)$ | information matrix |
| IID | independent and identically distributed |

**Acknowledgment (HEADING 5)**

The preferred spelling of the word "acknowledgment" in America is without an "e" after the "g". Avoid the stilted expression "one of us (R. B. G.) thanks …". Instead, try "R. B. G. thanks…". Put sponsor acknowledgments in the unnumbered footnote on the first page.

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