International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 5, 2025
https://theaspd.com/index.php

# Evaluating the Stability and Resilience of Ecosystems through the Structural Analysis of Fuzzy Super Subdivided Graphs

**T. Bharathi [1], Jeba Sherlin M [2], S. Shiny Paulin [3]**

[1]Assistant Professor, Department of Mathematics, Loyola College, University of Madras, Chennai, 600034, Tamil Nadu, India. drtbharathi2024@gmail.com. ORCID iD: 0000-0003-2460-2682.
[2] Research Scholar, Department of Mathematics, Loyola College, University of Madras, Chennai, 600034, Tamil Nadu, India. m.sherlinmohan@gmail.com. ORCID iD: 0009-0003-3058-3665.
[3]Research Scholar, Department of Mathematics, Loyola College, University of Madras, Chennai, 600034, Tamil Nadu, India. paulinshiny@gmail.com. ORCID iD: 0009-0006-1291-0484

## Abstract

*This paper investigates the structural properties of fuzzy super subdivided graphs and their role in ecosystem stability and resilience analysis. We introduce edge connectivity concepts, vertex and edge degree classifications, and subgraph structures in fuzzy super subdivided graphs. The study examines the interrelations between these properties and presents theoretical insights through formal definitions, theorems, and illustrations. Using a fuzzy super subdivision model, we analyze the stability and resilience of the ecosystem by modeling complex ecological interactions. The results demonstrate that fuzzy super subdivision graph provides a mathematical framework to assess ecosystem responses to disturbances and maintain ecological balance.*

***Keywords:*** *Super subdivision, edges, degree, induced, subgraphs*

*2020 Mathematics Subject Classification Codes: 05C72, 03B52, 94D05.*

## 1. Introduction

To play a crucial role in modeling and analyzing systems and to address uncertainty, Rosenfeld [1] proposed fuzzy graph theory, inspired by the work of L.A. Zadeh on fuzzy sets and fuzzy relations in 1965 [7]. Bhutani and Rosenfeld [6] defined strong edges in fuzzy graphs, where an edge is strong if its membership value equals the strength of connectedness between its vertices. In [5], the authors have also demonstrated the existence of a strong path between any two vertices in a fuzzy graph. Moderson and Peng [3] discussed the concept of strong fuzzy subgraphs and the operations of fuzzy graphs. Fuzzy graph theory and its extensions have found applications particularly significant in domains such as network analysis, decision- making, and many other fields to handle uncertainty or vagueness [2,11,12,14].

The study of fuzzy super subdivision graphs (FSSG) involves super subdividing each edge of the fuzzy graph in the context of fuzzy graph theory with a membership value between 0 and 1 [9]. This methodology represents flexibility and accuracy in relationships through connectivity. The comprehensive analysis of (FSSG) presented in this paper is motivated by the need to extend the theoretical framework and practical applications of (FSSG). This paper aims to explicitly discuss various aspects of fuzzy super subdivided graphs, which include the nature of edges connectivity, and subgraphs along with their degrees. Through detailed theorems and illustrations, we investigate vertices and edges, strength, strongness, connectedness, subgraphs and degrees of vertices and edges. One of the key theoretical findings is that the strength of the fuzzy super subdivided path between any two vertices of (FSSG) is found to be identical. Additionally, the relation between strength and strongness of paths and edges is explored. A strong fuzzy super subdivided path is the strongest $U - V$ fuzzy super subdivided path if it contains $\alpha$- strong and $\beta$- strong edges. The insight of the comparison between fuzzy super subdivided subgraphs is that every fuzzy super subdivided induced subgraph is a fuzzy super subdivided subgraph but the converse does not hold.

In addition, the novelty of this paper lies in the application of FSSG to the analysis of the stability and resilience of the ecosystem. This model enables a deeper understanding of the interactions, strength, and influential factors of species within ecosystems. By capturing the complexity, variability, and connectivity of ecological networks, this model offers valuable insights that can be applied to ecosystem management and conservation efforts. To determine the degree of connectivity and interaction with factors, the stability index is calculated based on the weighted degree centrality method.

The basic definitions are outlined in Section 2 as preliminaries, followed by discussions on the edge's connectivity of Section 3. Section 4 includes classifications of FSSG subgraphs and Section 5 includes the relationship between the vertex and edge degrees of FSSG, with definitions and theorems. Section 6 provides the detailed application on species interaction along with their factors.

## 2. Preliminaries

**Definition 2.1** [8] A fuzzy graph $G = (V, E, \sigma, \mu)$ corresponding to the crisp graph $G$ is a non-empty set $V$ together with a pair of functions $\sigma: V \to [0,1]$ and $\mu: E \to [0,1]$ such that for all $u, v \in V$, $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$, where $\sigma(u), \sigma(v)$ and $\mu(u, v)$ represent the membership values of the vertex $u$ and $v$ and $(u, v)$ is the corresponding adjacent edge respectively.

**Definition 2.2** [10] Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph. An edge $(u, v)$ is called strong in $G$ if $\mu(u, v) > 0$ and $\mu(u, v) \geq CONN_{G-(u,v)}(u, v)$.

**Definition 2.3** [9] Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph with $p$ vertices and $q$ edges. The fuzzy super subdivision graph is defined as $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ by replacing each edge $v_iv_j \in E$ such that $i$ G j and $1 \leq i, j \leq p$, by a complete bipartite graph $k_{2,m}$ for $m > 1$ in such a way that the ends of $v_iv_j$ are merged with the 2- vertices part of $k_{2,m}$. Here $V_{SS} = V \cup V^*$, where $V^*$contains super subdivided vertices $w_{(p-1)t}$ with $1 \leq (p-1) \leq q$ and $1 \leq t \leq m$ and $E_{SS}$ is the collection of super subdivided edges $e_{rs}$ with $1 \leq r \leq q$ and $1 \leq s \leq 2m$ satisfying the following conditions

i. $\sigma_{SS}(v_i) < \sigma_{SS}(w_{(p-1)t}) > \sigma_{SS}(v_j)$ ,
ii. $\mu_{SS}(v_i, w_{(p-1)t}) = \sigma_{SS}(v_i) \wedge \sigma_{SS}(w_{(p-1)t})$ where $v_i, v_j \in V$ and $w_{(p-1)t} \in V_{ss}$.

**Definition 2.4** [9] Let $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ be the fuzzy super subdivided graph. The degree of a vertex $v$ is defined by the sum of the membership value of super subdivided edges incident with $v$ and it is denoted by $d_{SS}(v) = \sum_{uGv} \mu_{SS}(v_i, w_{(p-1)t})$. The minimum and the maximum degree is defined by $\delta_{SS}(G) = \wedge\{d_{SS}(w_{(p-1)t}) \mid w_{(p-1)t} \in V_{SS}\}$ and $\Delta_{SS}(G) = \vee\{d_{SS}(w_{(p-1)t}) \mid w_{(p-1)t} \in V_{SS}\}$ respectively.

**Definition 2.5** [8] The fuzzy super subdivided graph $SS_f(G)$ is said to be a complete fuzzy super subdivided graph $C_f(G)$ if $\mu_{SS}(v_i, w_{(p-1)t}) = min\{\sigma_{SS}(v_i), \sigma_{SS}(w_{(p-1)t})\}$ where $v_i \in V$ and $w_{(p-1)t} \in V_{ss}$ for $1 \leq i \leq p$ , $1 \leq p \leq q$ and $1 \leq t \leq m$.

## 3. Edge Connectivity

**Definition 3.1.** The edge $(v_i, w_{(p-1)t})$ in FSSG is a strong edge if $\mu_{SS}(v_i, w_{(p-1)t}) > 0$ and $\mu_{SS}(v_i, w_{(p-1)t}) \geq CONN_{SS_f(G)-(v_i,w_{(p-1)t})}(v_i, w_{(p-1)t})$.

**Definition 3.2.** The edge $(v_i, w_{(p-1)t})$ in FSSG is an effective edge if $\mu_{SS}(v_i, w_{(p-1)t}) = \sigma_{SS}(v_i) \wedge \sigma_{SS}(w_{(p-1)t})$. The vertices $v_i$ and $w_{(p-1)t}$ are called effective neighbors. The set of all effective neighbors of $v_i$ is called effective neighborhood of $v_i$.

**Definition 3.3.** [13] Depending on the $CONN_{SS_f(G)}(v_i, w_{(p-1)t})$ of an edge $(v_i, w_{(p-1)t})$ in $SS_f(G)$, the edges are classified as follows:

i. If an edge $(v_i, w_{(p-1)t})$ is $\alpha$- strong then, $\mu_{SS}(v_i, w_{(p-1)t}) > CONN_{SS_f(G)-(v_i,w_{(p-1)t})}(v_i, w_{(p-1)t})$

International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 5, 2025
https://theaspd.com/index.php

ii. If an edge $(v_i, w_{(p-1)t})$ is $\beta$- strong then, $\mu_{SS}(v_i, w_{(p-1)t}) = CONN_{SS_f(G)-(v_i,w_{(p-1)t})}(v_i, w_{(p-1)t})$

iii. If an edge $(v_i, w_{(p-1)t})$ is ð- arc then, $\mu_{SS}(v_i, w_{(p-1)t}) < CONN_{SS_f(G)-(v_i,w_{(p-1)t})}(v_i, w_{(p-1)t})$.

**Example 3.4.** Consider fuzzy super subdivided square graph $SS_f(S_4)$ in Figure 1.

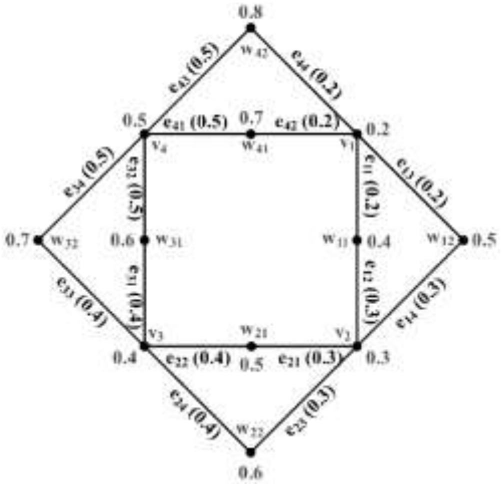



**Figure 1.** Fuzzy super subdivided square graph $SS_f(S_4)$ for $k_{2,2}$

From the above Figure 1, every $(v_i, w_{(p-1)t})$ edge in $SS_f(S_4)$ has more than two paths. The strength of connectedness for all the edges are given below:

For $v_1$ and $w_{11}$,

$$\mu_{SS}(v_1, w_{11}) = 0.2 \text{ and } CONN_{SS_f(S_4)-(v_1,w_{11})}(v_1, w_{11}) = 0.2$$

Therefore,

$$\mu_{SS}(v_1, w_{11}) = CONN_{SS_f(S_4)-(v_1,w_{11})}(v_1, w_{11})$$

Similarly for the vertex $v_1$ and $w_{12}$. Therefore, the $\beta$- strong edges are: $(v_1, w_{11})$, $(v_1, w_{12})$, $(v_2, w_{21})$, $(v_2, w_{22})$, $(v_3, w_{31})$, $(v_3, w_{32})$, $(v_1, w_{41})$, $(v_1, w_{22})$.

For $v_2$ and $w_{11}$,

$$\mu_{SS}(v_2, w_{11}) = 0.3 \text{ and } CONN_{SS_f(S_4)-(v_2,w_{11})}(v_2, w_{11}) = 0.2$$

Therefore,

$$\mu_{SS}(v_2, w_{11}) > CONN_{SS_f(S_4)-(v_2,w_{11})}(v_2, w_{11})$$

Similarly for the vertex $v_2$ and $w_{12}$. Therefore, the $\alpha$ – strong edges are:
$(v_2, w_{11})$, $(v_1, w_{12})$, $(v_3, w_{21})$, $(v_3, w_{22}$ z, $(v_4, w_{31})$, $(v_4, w_{32})$, $(v_4, w_{41})$, $(v_4, w_{22})$.

**Lemma 3.5.** All the $E_{SS}$ edges of fuzzy super subdivided graphs are effective edges.

**Proof.** Let us consider an arbitrary edge $e = (v_i, w_{(p-1)t})$ in fuzzy super subdivision graph $SS_f(G)$ which connects vertices $v_i$ and $w_{(p-1)t}$. According to the Definition 2.3, the membership value of an arbitrary edge $e$ is given by

$$\mu_{SS}(v_i, w_{(p-1)t}) = \sigma_{SS}(v_i) \wedge \sigma_{SS}(w_{(p-1)t}),$$

where $v_i \in V$ and $w_{(p-1)t} \in V_{ss}$. Given that every FSSG is a complete FSSG, every edge satisfies this condition. Therefore, the arbitrary edge $e$ in FSSG forms an effective edge. This holds true for all FSSG, providing every edge in the graph is effective.

**Lemma 3.6.** All the $E_{SS}$ edges of fuzzy super subdivided graphs are strong edges.

**Proof.** Given an arbitrary edge $e = (v_i, w_{(p-1)t})$ in fuzzy super subdivision graph $SS_f(G)$ where $v_i \in V$ and $w_{(p-1)t} \in V_{ss}$. By Definition 2.3, the membership value of an arbitrary edge $e$ and the corresponding vertices are given by

$$\sigma_{SS}(v_i) < \sigma_{SS}(w_{(p-1)t}) > \sigma_{SS}(v_j) \text{ and } \mu_{SS}(v_i, w_{(p-1)t}) = \sigma_{SS}(v_i) \wedge \sigma_{SS}(w_{(p-1)t})$$

On satisfying these conditions, $\mu_{SS}(v_i, w_{(p-1)t})$ must be greater than zero. Referring to Example 3.4, $CONN_{SS_f(G)-(v_i,w_{(p-1)t})}(v_i, w_{(p-1)t})$ is either less than or equal to $\mu_{SS}(v_i, w_{(p-1)t})$, satisfying the condition of strong edge. Thus, the arbitrary edge $e$ in FSSG satisfies for strong edge, concluding that all edges are strong edges.

**Theorem 3.7.** A fuzzy super subdivided path has both $\alpha$- strong and $\beta$ − strong edges if it is a strong fuzzy super subdivided path.

**Proof.** Consider a fuzzy super subdivided path from $U$ to $V$ with a sequence of $(v_i, w_{(p-1)t})$ edges, $1 \le i, j \le p$, $1 \le (p-1) \le q$ and $1 \le t \le m$. To prove by contradiction, assume there exist a path that contains both $\alpha$- strong and $\beta$ − strong edges but is not a strong fuzzy super subdivided path. This implies that there exists at least one edge in the path that is not strong. However, By Lemma 3.6, every edge $e_{rs}$ in the fuzzy super subdivided path with both α-strong and $\beta$ − strong edges are strong. This contradicts our assumption, follows that any path containing both α-strong and $\beta$ − strong edges must be strong.

Conversely, assume there exists a strong fuzzy super subdivided path from $U$ to $V$ in $SS_f(G)$ does not contain both $\alpha$-strong and $\beta$ − strong edges.

**Case (i):** Suppose there are no $\alpha$- strong edges in a strong fuzzy super subdivided path, then the membership value for each edge $\mu_{SS}(v_i, w_{(p-1)t})$, must be less than the strength of connectedness $CONN_{SS_f(G)}(v_i, w_{(p-1)t})$. However, this contradicts the definition of a strong fuzzy super subdivided path, which requires all edges to be strong.

**Case (ii):** If there are no $\beta$- strong edges in a strong fuzzy super subdivided path, then the membership value for each edge $\mu_{SS}(v_i, w_{(p-1)t})$, must be less than or greater than the strength of connectedness $CONN_{SS_f(G)}(v_i, w_{(p-1)t})$. Again, this contradicts the definition of a strong fuzzy super subdivided path. Since both cases leads to contradictions, the assumption that the strong fuzzy super subdivided path does not contain both $\alpha$- strong and $\beta$ − strong edges do not hold true.

**Proposition 3.8.** A strong fuzzy super subdivided path from $U$ to $V$ is the strongest $U$-$V$ fuzzy super subdivided path if it contains $\alpha$- strong and $\beta$ − strong edges.

**Proof.** Let $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ be the fuzzy super subdivided graph. Let $A$ be a strong fuzzy super subdivided $U - V$ path in $SS_f(G)$, consist of both $\alpha$- strong and $\beta$ − strong edges. Assume by contradiction that $A$ is not the strongest $U$-$V$ FSSG path. Now consider another strongest path $B$ in $SS_f(G)$. Then, the union of path $A \cup B$ will contain at least one cycle $C$. Since $A$ contains both $\alpha$- strong and $\beta -$ strong edges, every edge of $C - A$ path will have a strength equal to the strength of $A$. Let $(u, v)$ be the weakest edge of path $C$ which is also an edge of path $A$. Let $C'$ be the $u - v$ path in $C$, not containing the edge $(u, v)$. Then,

$$\mu(u, v) \leq \text{Strength of } C' \leq CONN_{SS_f(G)-(u,v)}(u, v)$$

This contradicts our assumption that $(u, v)$ is not $\alpha$- strong and $\beta -$ strong edge to be the strongest path. Therefore, $A$ is the strongest $U$-$V$ fuzzy super subdivided path containing both $\alpha$- strong and $\beta -$ strong edges.

**Theorem 3.9.** $(v_i, w_{(p-1)t})$ is $\beta -$ strong, provided $(v_i, w_{(p-1)t})$ is the weakest edge of all the strongest paths.

**Proof.** Let $A$ be the strongest path in FSSG, $SS_f(G)$. Consider $(v_i, w_{(p-1)t})$ be the weakest edge in the strongest path $A$. By contrary, assume that $(v_i, w_{(p-1)t})$ is not $\beta -$ strong edge in the strongest path. Now, removing the weakest edge $(v_i, w_{(p-1)t})$ from the strongest path. Then the strength of connectedness of $SS_f(G) - (v_i, w_{(p-1)t})$ is given by

$$\mu_{SS}(v_i, w_{(p-1)t}) \leq CONN_{SS_f(G)-(v_i,w_{(p-1)t})}(v_i, w_{(p-1)t})$$

Suppose $(v_i, w_{(p-1)t})$ is not $\beta -$ strong, then

$$\mu_{SS}(v_i, w_{(p-1)t}) \text{ G } CONN_{SS_f(G)-(v_i,w_{(p-1)t})}(v_i, w_{(p-1)t})$$

This implies,

$$\mu_{SS}(v_i, w_{(p-1)t}) < CONN_{SS_f(G)-(v_i,w_{(p-1)t})}(v_i, w_{(p-1)t})$$

However, the removal of the weakest edge does not decrease the strength of connectedness and remains the strongest path. Then we have,

$$\mu_{SS}(v_i, w_{(p-1)t}) = CONN_{SS_f(G)-(v_i,w_{(p-1)t})}(v_i, w_{(p-1)t}),$$

which is a contradiction. Therefore, $(v_i, w_{(p-1)t})$ must be $\beta -$ strong, provided it is the weakest edge in the strongest path.

**Corollary 3.10.** ð $-$ arc does not exist for $m \geq 2$, where $m$ is number of copies of super subdivision.

**Proof.** By Definition 2.5, FSSG graph is a complete FSSG graph. Complete fuzzy super subdivided graph does not contain ð $-$ arc edges. This proves this corollary.

**Theorem 3.11.** Let $SS_f(G)$ be the fuzzy super subdivided graph, then the number of $\alpha$- strong and $\beta -$ strong edges are as follows

i. $m(n - 1)$ for any fuzzy super subdivided path graph
ii. $mn$ for any fuzzy super subdivided cycle graph
iii. $m(n - 1)$ for any fuzzy super subdivided tree graph, where $n$ is the number of vertices of the given fuzzy graph and $m$ copies of super subdivision.

**Proof.** Let $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ be the fuzzy super subdivided graph with $m$ copies of super subdivision. The number of $\alpha$- strong and $\beta -$ strong edges in fuzzy super subdivided graph are proved as follows:

**Case (i).** Each vertex $v_i$ in the path contributes $(m - 1)$ $\alpha$- strong edges towards each of the $m$ vertices in the corresponding subdivision. Similarly, each vertex contributes $(m - 1)$ β-strong edges towards each of the $m$ vertices in the corresponding super subdivision. Therefore, the total number of $\alpha$-strong and $\beta$-strong edges contributed by each vertex in the fuzzy super subdivided path is $(m - 1)m$. Since there are $n$ vertices in the given fuzzy path, the total number of $\alpha -$strong and $\beta -$strong edges in the fuzzy super subdivided path graph are $n \times m(m - 1)$. Thus, for $m$ subdivisions per edge and $n$ vertices in the path, the total number of $\alpha$-strong and $\beta$-strong edges in the fuzzy super subdivided path graph are $m(n - 1)$.

**Case (ii).** In a fuzzy super subdivided cycle graph, each vertex contributes the same number of $\alpha$ −strong and $\beta$ −strong edges as in the fuzzy super subdivided path graph. However, since there are no end vertices in a cycle graph, all vertices contribute equally. Therefore, for a given fuzzy cycle with $n$ vertices, each vertex contributes $m$ $\alpha$-strong and $\beta$-strong edges in $SS_f(G)$. Since there are $n$ vertices in the cycle, the total number of $\alpha$-strong and $\beta$-strong edges in the fuzzy super subdivided cycle graph are $n \times m$. Thus, for $m$ subdivisions per edge and $n$ vertices in the cycle, the total number of $\alpha$-strong and $\beta$-strong edges in the fuzzy super subdivided cycle graph are $nm$.

**Case (iii).** In a fuzzy super subdivided tree graph, each vertex contributes α-strong and β-strong edges differently based on whether it is an internal vertex or a pendent vertex. Each internal vertex contributes $m$ $\alpha$ −strong and $\beta$-strong edges as it is connected to $m$ super subdivisions. For each pendent vertex, it contributes only $(n-1)$ $\alpha$-strong and $\beta$-strong edges towards its $m$ super subdivisions. Thus, the total number of $\alpha$-strong and $\beta$-strong edges in the fuzzy super subdivided tree graph are $m(n-1)$.

**Remark 3.12.** For any fuzzy super subdivided graph, the number of $\alpha$- strong edges are equal to the number of $\beta$-strong edges.

**4. Subgraphs**

**Definition 4.1.** Let $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ be the fuzzy super subdivided graph. $SS_f(H) = (P_{SS}, Q_{SS}, \tau_{SS}, \rho_{SS})$ is said to be the fuzzy super subdivided subgraph of $SS_f(G)$ if $P_{SS} \subseteq V_{SS}$ and $Q_{SS} \subseteq E_{SS}$ such that $\sigma_{SS}(v_i) = \tau_{SS}(v_i)$, $\forall\, v_i \in V_{SS}$ and $\mu_{SS}(v_i, w_{(p-1)t}) = \rho_{SS}(v_i, w_{(p-1)t})$, $\forall\, (v_i, w_{(p-1)t}) \in E_{SS}$.

**Definition 4.2.** $SS_f(H) = (P_{SS}, Q_{SS})$ is said to be spanned if it is induced by eliminating $E_{SS}$ edge or edges in $SS_f(G)$ is called FSSD spanning subgraph such that $P_{SS} = V_{SS}$ , $Q_{SS} \subseteq E_{SS}$ where $\sigma_{SS}(v_i) = \tau_{SS}(v_i)$ , $\forall\, v_i \in V_{SS}$ and $\mu_{SS}(v_i, w_{(p-1)t}) = \rho_{SS}(v_i, w_{(p-1)t})$, $\forall\, (v_i, w_{(p-1)t}) \in E_{SS}$.

**Definition 4.3.** $SS_f(H) = (P_{SS}, Q_{SS})$ is said to be the fuzzy super subdivided bridge subgraph of $SS_f(G)$, if the removal of $E_{SS}$ edge or edges in $SS_f(G)$ disconnects the graph, such that $P_{SS} \subseteq V_{SS}$, $Q_{SS} \subseteq E_{SS}$ where $\sigma_{SS}(v_i) = \tau_{SS}(v_i)$, $\forall\, v_i \in V_{SS}$ and $\mu_{SS}(v_i, w_{(p-1)t}) = \rho_{SS}(v_i, w_{(p-1)t})$, $\forall\, (v_i, w_{(p-1)t}) \in E_{SS}$.

**Definition 4.4. Let** $SS_f(H) = (P_{SS}, Q_{SS}, \tau_{SS}, \rho_{SS})$ be the fuzzy super subdivided induced subgraph of $SS_f(G)$, if it is induced by eliminating only super subdivided vertex or vertices $(w_{(p-1)t})$ of $V_{SS}$ in $SS_f(G)$ with $\sigma_{SS}(v_i) = \tau_{SS}(v_i)$, $\forall\, v_i \in V_{SS}$ and $\mu_{SS}(v_i, w_{(p-1)t}) = \rho_{SS}(v_i, w_{(p-1)t})$, $\forall\, (v_i, w_{(p-1)t}) \in E_{SS}$.

**Definition 4.5.** Let $P_{SS} < V_{SS}$ be the vertex subset of $SS_f(G)$. $SS_f[P_{SS}]$ is said to be the FSSD vertex induced subgraph of $SS_f(G)$ with the vertex having maximum degree and vertices adjacent to it such that $\sigma_{SS}(v_i) = \tau_{SS}(v_i)$, $\forall\, v_i \in V_{SS}$ and $\mu_{SS}(v_i, w_{(p-1)t}) = \rho_{SS}(v_i, w_{(p-1)t})$, $\forall\, (v_i, w_{(p-1)t}) \in E_{SS}$.

**Definition 4.6.** Let $Q_{SS} < V_{SS}$ be the edge subset of $SS_f(G)$. $SS_f[Q_{SS}]$ is said to be the FSSD edge induced subgraph of $SS_f(G)$ with the edge having maximum degree along with the edges incident to its end vertices such that $\sigma_{SS}(v_i) = \tau_{SS}(v_i)$, $\forall\, v_i \in V_{SS}$ and $\mu_{SS}(v_i, w_{(p-1)t}) = \rho_{SS}(v_i, w_{(p-1)t})$, $\forall\, (v_i, w_{(p-1)t}) \in E_{SS}$.

**Definition 4.7.** $SS_f(G)$ is said to be full fuzzy super subdivided subgraph if $\sigma_{SS}(v_i) > 0$ $\forall\, v_i \in V_{SS}$ and $\mu_{SS}(v_i, w_{(p-1)t}) > 0$ $\forall\, (v_i, w_{(p-1)t}) \in E_{SS}$.

**Definition 4.8.** $SS_f(G)$ is said to be maximum spanning fuzzy super subdivided subgraph if the sum of the edge membership values of connected spanning subgraph is maximum.

**Example 4.9.** Consider fuzzy super subdivision of tadpole graph $SS_f(T_4)$ in Figure 2. The fuzzy super subdivided spanning subgraph in Figure 3 is obtained by removing $E_{SS}$ edges: $e_{12}$, $e_{13}$, $e_{23}$, $e_{32}$, $e_{41}$ . The fuzzy super subdivided induced subgraph in Figure 4 is obtained by the removal of the super subdivided edges: $e_{12}$, $e_{13}$, $e_{22}$, $e_{24}$, $e_{42}$, $e_{44}$, $e_{34}$, $e_{32}$. The fuzzy super subdivided induced subgraph in Figure 5 is obtained by the removal of the super subdivided vertices: $w_{12}$ , $w_{21}$ , $w_{41}$ . The fuzzy super subdivided vertex induced subgraph in Figure 6 is obtained with the vertex $v_3$ having maximum degree and its adjacent vertices with edges and fuzzy super subdivided edge induced subgraph in Figure 7 is obtained with the edges $e_{41}$ and $e_{43}$ having maximum degree and vertices adjacent to its end vertices.

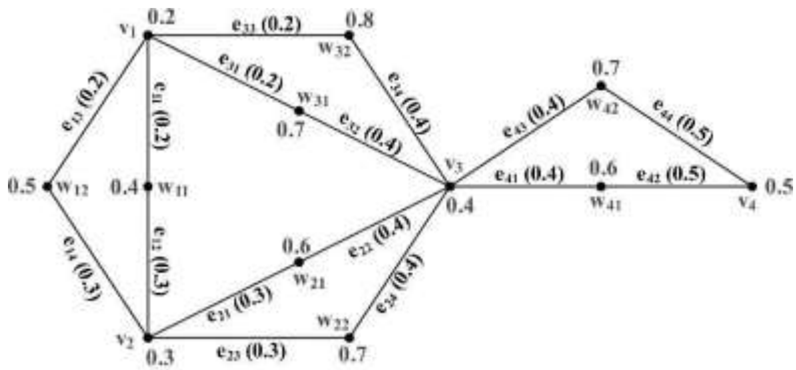



**Figure 2.** Fuzzy super subdivision of tadpole graph $SS_f(T_4)$ for $k_{2,2}$

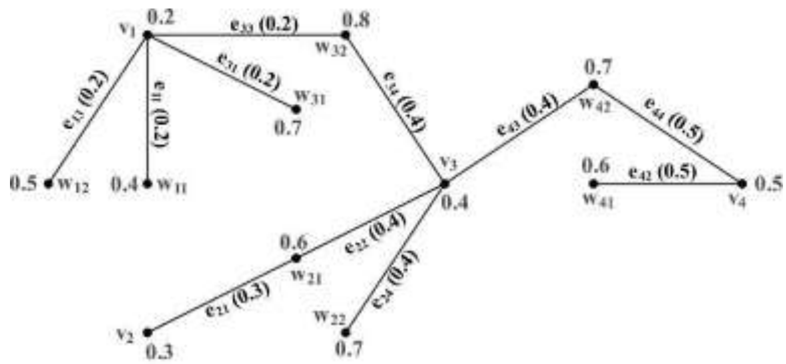



**Figure 3.** Fuzzy super subdivided spanning subgraph of tadpole graph $SS_f(T_4)$ for $k_{2,2}$

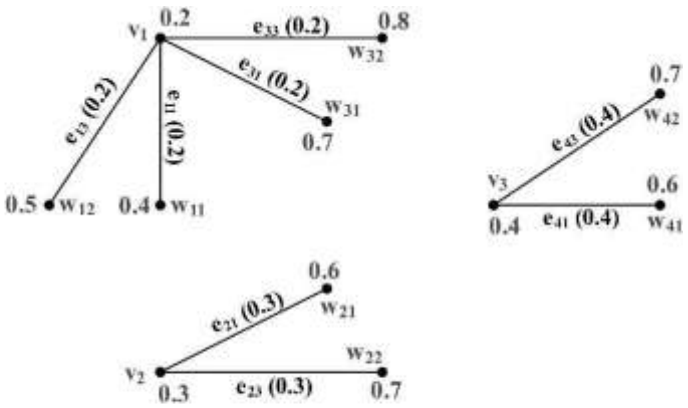



**Figure 4.** Fuzzy super subdivided bridge subgraph of tadpole graph $SS_f(T_4)$ for $k_{2,2}$

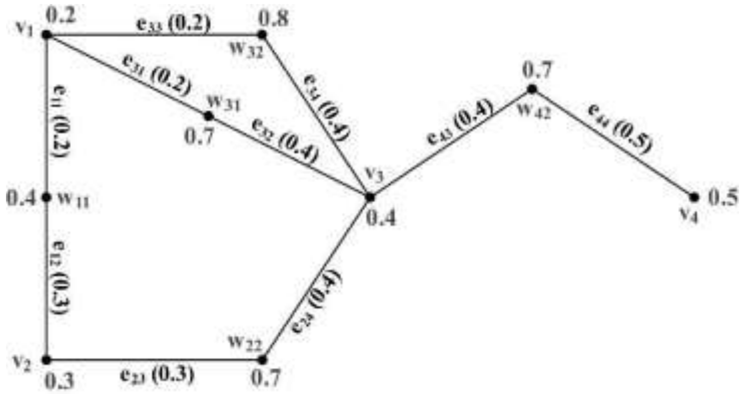



**Figure 5.** Fuzzy super subdivided induced subgraph of tadpole graph $SS_f(T_4)$ for $k_{2,2}$

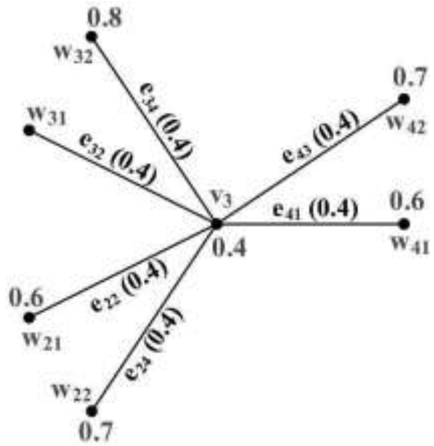



**Figure 6.** Fuzzy super subdivided vertex-induced subgraph of tadpole graph $SS_f(T_4)$ for $k_{2,2}$

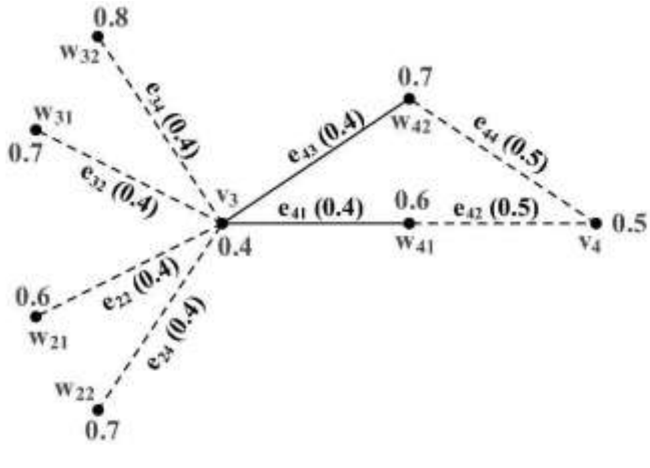


**Figure 7.** Fuzzy super subdivided edge-induced subgraph of tadpole graph $SS_f(T_4)$ for $k_{2,2}$

**Theorem 4.10.** Every fuzzy super subdivided induced subgraph is a fuzzy super subdivided subgraph but the converse is not true.

**Proof.** Consider an induced subgraph $SS_f(H) = (P_{SS}, Q_{SS})$ of $SS_f(G)$ which includes $P_{SS} \subseteq V_{SS}$ and $Q_{SS} \subseteq E_{SS}$. By Definition 4.4,

$$\sigma_{SS}(v_i) = \tau_{SS}(v_i),\ \forall\ v_i \in V_{SS} \text{ and } \mu_{SS}(v_i, w_{(p-1)t}) = \rho_{SS}(v_i, w_{(p-1)t}),\ \forall\ (v_i, w_{(p-1)t}) \in E_{SS}.$$

Consider any of the super subdivided $V_{SS}$ vertices $\{w_{11}, w_{12}, \dots, w_{1t}, w_{21}, \dots, w_{2t}, \dots, w_{(p-1)1}, \dots w_{(p-1)t}\}$ for the removal. Since $SS_f(H)$ is obtained by eliminating only certain $(w_{(p-1)t})$ vertices of $V_{SS}$ in $SS_f(G)$, it follows that $SS_f(H)$ maintains the structural properties of a FSSG, as one of the end points of edges $\{(v_1, w_{11}), \dots, (v_p, w_{(p-1)t})\}$ are included in $P_{SS}$ of $SS_f(H)$. Therefore, every fuzzy super subdivided induced subgraph is a fuzzy super subdivided subgraph.

Conversely, consider a fuzzy super subdivided subgraph $SS_f(H) = (P_{SS}, Q_{SS})$ of $SS_f(G)$ that is obtained by selecting a subset of vertices $P_{SS} \subseteq V_{SS}$ and edges $Q_{SS} \subseteq E_{SS}$ from $SS_f(G)$. $SS_f(H)$ may not maintain the property of being induced, as not all edges between the selected vertices in $P_{SS}$ are necessarily included. If $SS_f(H)$ contains only a subset of $V_{SS} -$ vertices and $Q_{SS} -$ edges, but not all edges between $P_{SS}$ vertices, it would not be considered an induced subgraph. Therefore, $SS_f(H)$ may be a fuzzy super subdivided subgraph, it may not necessarily be an induced subgraph. Every fuzzy super subdivided induced subgraph is a fuzzy super subdivided subgraph, but the converse is not true.

**Theorem 4.11** In any vertex-induced subgraph of a fuzzy super subdivided graph, a subgraph consists of one point union of $m$ pendent edges of $SS_f(G)$.

**Proof.** Let $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ be the fuzzy super subdivided graph. Consider a vertex-induced subgraph $SS_f(H) = (P_{SS}, Q_{SS}, \tau_{SS}, \rho_{SS})$ of $SS_f(G)$, obtained by selecting a subset of vertices $P_{SS} \subseteq V_{SS}$, with the condition of vertex having maximum degree and the vertices adjacent to its end vertices including edges incident to those vertices. Since $SS_f(H)$ is a vertex-induced subgraph, it contains all edges incident to the selected vertices in $P_{SS}$, but no additional edges. Each vertex in $P_{SS}$ corresponds to a super subdivided vertex $w_{(p-1)t}$ in $SS_f(G)$. The edges incident to each vertex $w_{(p-1)t}$ in $SS_f(G)$ form a pendent edge structure, where each edge is connected to $w_{(p-1)t}$. Since $SS_f(H)$ includes all edges incident to the selected vertices in $P_{SS}$, it consists of one point union of $m$ pendent edges associated with each selected vertex. Therefore, in any FSSD vertex-induced subgraph, the subgraph consists of one point union of $m$ pendent edges of $SS_f(G)$.

**Proposition 4.12.** Any edge in $SS_f(G)$ is strong, if it is an edge of at least one maximum spanning subgraph.

**Proof.** Consider $SS_f(H) = (P_{SS}, Q_{SS})$ be the fuzzy super subdivided spanning subgraph with the connected $H_1, H_2, H_3$ components. Assume there exists an edge $(v_i, w_{(p-1)t})$ in $SS_f(H)$ in maximum spanning subgraph but not a strong edge. This implies that

$$\mu_{SS}(v_i, w_{(p-1)t})\ \text{G}\ CONN_{SS_f(G)-(v_i, w_{(p-1)t})}(v_i, w_{(p-1)t})$$

Since this edge is not a strong edge, there exists at least one other strong edge $(v_j, w_{pt}) \in E_{ss}$ with the maximum membership value. Then $(v_i, w_{(p-1)t}) \in E_{ss}$ is replaced with $(v_j, w_{pt}) \in E_{ss}$ in the maximum spanning subgraph, implying $\mu_{SS}(v_j, w_{pt}) > \mu_{SS}(v_i, w_{(p-1)t})$. Now, the sum of the edge membership value gets increased. Therefore, $\mu_{SS}(v_i, w_{(p-1)t})$ is not a part of at least one maximum spanning subgraph, the sum of the edge membership values of the connected spanning subgraph that includes $(v_j, w_{pt})$ is maximum. This contradicts our assumption that $(v_i, w_{(p-1)t})$ in $SS_f(H)$ in maximum spanning subgraph. Hence, any edge in maximum fuzzy super subdivided spanning subgraph must be a strong edge.

## 5. Degree of a vertex and an edge

**Definition 5.1.** Let $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ be the fuzzy super subdivided graph. The effective degree of a vertex $v_i$ is the sum of the membership value of the effective edges' incident at $v_i$ and denoted by $d_{E(SS)}(v_i)$. The minimum and the maximum effective degree is defined by $\delta_{E(SS)}(G) = \wedge\{d_{E(SS)}(v_i) : v_i \in V_{SS}\}$ and $\Delta_{E(SS)}(G) = \vee\{d_{E(SS)}(v_i) : v_i \in V_{SS}\}$ respectively. The strong degree of a vertex, $d_{S(SS)}(v_i)$ is the sum of membership value of all the strong edges adjacent to the vertex $v_i$.

**Definition 5.2.** The neighborhood degree of a vertex $v_i$ is the sum of membership value of all the vertices adjacent to the vertex $v_i$ and denoted by $d_{N(SS)}(v_i)$. The effective neighborhood degree of a vertex $v_i$ is the sum of membership value of all the vertices of the effective edges adjacent to the vertex $v_i$ and denoted by $d_{EN(SS)}(v_i)$. The strong neighborhood degree of a vertex $v_i$ is the sum of membership value of all the vertices of the strong edges adjacent to the vertex $v_i$ and denoted by $d_{SN(SS)}(v_i)$.

**Definition 5.3.** The degree of an edge $(v_i, w_{(p-1)t}) \in E_{SS}$ of $SS_f(G)$ is defined as $D_{e(SS)}(v_i, w_{(p-1)t}) = d_{SS}(v_i) + d_{SS}(w_{(p-1)t}) - 2\mu_{SS}(v_i, w_{(p-1)t}) \ \forall \ v_i \in V_{SS}$ and $(v_i, w_{(p-1)t}) \in E_{SS}$.

**Definition 5.4.** The total degree of an edge $(v_i, w_{(p-1)t}) \in E_{SS}$ of $SS_f(G)$ is defined as $TD_{e(SS)}(v_i, w_{(p-1)t}) = D_{e(SS)}(v_i, w_{(p-1)t}) + \mu_{SS}(v_i, w_{(p-1)t}) \ \forall \ v_i \in V_{SS}$ and $(v_i, w_{(p-1)t}) \in E_{SS}$.

**Example 5.5.** Consider fuzzy super subdivision of Butterfly graph $SS_f(B_2)$ in Figure 8. Degree of a vertex, Effective degree and strong degree of a vertex is given in Table 1.

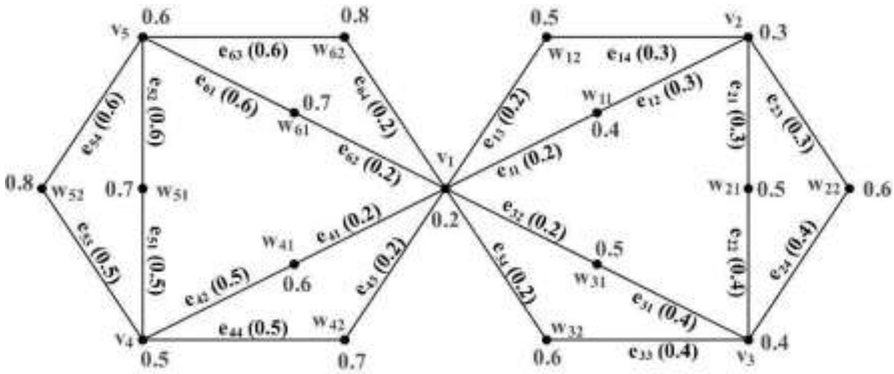



**Figure 8.** Fuzzy super subdivision of butterfly graph $SS_f(B_2)$ for $k_{2,2}$

| Vertices of $SS_f(B_2)$ | $d_{SS}(v_i)$ = $d_{E(SS)}(v_i)$ = $d_{S(SS)}(v_i)$ | $d_{N(SS)}(v_i)$ = $d_{EN(SS)}(v_i)$ = $d_{SN(SS)}(v_i)$ |
|---|---|---|
| $d_{SS}(v_1)$ | 1.6 | 4.8 |
| $d_{SS}(w_{11})$ , $d_{SS}(w_{12})$ | 0.5 | 0.5 |
| $d_{SS}(v_2)$ | 1.2 | 2.0 |
| $d_{SS}(w_{21})$ , $d_{SS}(w_{22})$ | 0.7 | 0.7 |
| $d_{SS}(v_3)$ | 1.6 | 2.2 |
| $d_{SS}(w_{31})$ , $d_{SS}(w_{32})$ | 0.6 | 0.6 |
| $d_{SS}(v_4)$ | 2 | 2.8 |
| $d_{SS}(w_{41})$, $d_{SS}(w_{42})$ | 0.7 | 0.7 |
| $d_{SS}(w_{51})$, $d_{SS}(w_{52})$ | 1.1 | 1.1 |
| $d_{SS}(v_5)$ | 2.4 | 3.0 |
| $d_{SS}(w_{61})$, $d_{SS}(w_{62})$ | 0.8 | 0.8 |

**Table 1.** Degree, effective degree, strong degree, neighborhood degree, effective neighborhood and strong neighborhood degree of a vertex of fuzzy super subdivided Butterfly graph $SS_f(B_2)$

| Edges of $SS_f(B_2)$ | $D_{e(SS)}(v_i, w_{(p-1)t})$ | $TD_{e(SS)}(v_i, w_{(p-1)t})$ |
|---|---|---|
| $d_{SS}(v_1, w_{11})$, $d_{SS}(v_1, w_{12})$ | 1.7 | 1.9 |
| $d_{SS}(v_2, w_{11})$, $d_{SS}(v_2, w_{12})$ | 1.1 | 1.4 |
| $d_{SS}(v_2, w_{21})$, $d_{SS}(v_2, w_{22})$ | 1.3 | 1.6 |
| $d_{SS}(v_3, w_{21})$, $d_{SS}(v_3, w_{22})$ | 1.5 | 1.9 |
| $d_{SS}(v_3, w_{31})$, $d_{SS}(v_3, w_{32})$ | 1.4 | 1.8 |
| $d_{SS}(v_1, w_{31})$, $d_{SS}(v_1, w_{32})$ | 1.8 | 2.0 |
| $d_{SS}(v_1, w_{41})$, $d_{SS}(v_1, w_{42})$ | 1.9 | 2.1 |
| $d_{SS}(v_4, w_{41})$, $d_{SS}(v_4, w_{42})$ | 1.7 | 2.2 |
| $d_{SS}(v_4, w_{51})$, $d_{SS}(v_4, w_{52})$ | 2.1 | 2.6 |
| $d_{SS}(v_5, w_{51})$, $d_{SS}(v_5, w_{52})$ | 2.3 | 2.9 |
| $d_{SS}(v_5, w_{61})$, $d_{SS}(v_5, w_{62})$ | 2.0 | 2.2 |
| $d_{SS}(v_1, w_{61})$, $d_{SS}(v_1, w_{62})$ | 2.0 | 2.2 |

**Table 2.** Degree and Total degree of an edge of fuzzy super subdivided butterfly graph $SS_f(B_2)$

**Theorem 5.6.** As $SS_f(G)$ is a complete fuzzy super subdivided graph, every vertex has the same effective degree as its strong degree.

**Proof** Given that $SS_f(G)$ = $(V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ is a complete fuzzy super subdivided graph. Since $SS_f(G)$ is a complete FSSG (by Definition 2.5), every vertex is connected to all other vertices via effective edges. Then the effective degree of each vertex $v_i$ is given by

$$d_{E(SS)}(v_i) = \sum_{\forall v_i} \mu_{E(SS)}(v_i, w_{(p-1)t}) \quad \text{-} (5.1)$$

Also, every edge $(v_i, w_{(p-1)t})$ of FSSG is a strong edge. Then for each vertex $v_i$,

$$d_{S(SS)}(v_i) = \sum_{\forall v_i} \mu_{S(SS)}(v_i, w_{(p-1)t}) \quad \text{-} (5.2)$$

Since all the edges of FSSG are strong, effective degree includes all the membership values of strong edges. From (5.1) and (5.2), for each vertex $v_i$,

$$\sum_{\forall v_i} \mu_{E(SS)}(v_i, w_{(p-1)t}) = \sum_{\forall v_i} \mu_{S(SS)}(v_i, w_{(p-1)t})$$

Therefore,

$$d_{E(SS)}(v_i) = d_{S(SS)}(v_i).$$

**Corollary 5.7** Every vertex admits effective degree and strong degree that are equivalent to the degree of the vertex if $SS_f(G)$ is a complete fuzzy super subdivided graph.

**Proof.** From Theorem 5.6, each vertex $v_i$ has effective degree and strong degree which is equivalent to the degree of a vertex in $SS_f(G)$ (Refer table 1).

**Proposition 5.8** For any $SS_f(G)$, effective degree of a vertex is less than the neighborhood degree of a vertex.

**Proof.** Let $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ be a fuzzy super subdivided graph. Consider an arbitrary vertex $v_i \in V_{SS}$ in $SS_f(G)$. Since $SS_f(G)$ is a complete fuzzy super subdivided graph, every vertex is connected to all other vertices via effective edges/strong edges. Therefore, the sum of the membership values of the effective edges' incident at $v_i$ is less than the sum of the membership values of all the vertices adjacent to $v_i$. The neighborhood degree includes all vertices adjacent to $v_i$ and not just the effective edges. However, since $SS_f(G)$ is complete, the neighborhood degree of $v_i$ includes the membership values of all vertices adjacent to $v_i$. Hence, the sum of the membership values of the effective edges' incident at $v_i$ is strictly less than the sum of the membership values of all the vertices adjacent to $v_i$. Thus, the effective degree of a vertex is less than the neighborhood degree of the vertex in $SS_f(G)$.

**Theorem 5.9.** The effective, strong and neighborhood degrees of the super subdivided vertices $w_{(p-1)t}$ admits uniformity.

**Proof.** Let $w_{(p-1)t}$ be a super subdivided vertex of FSSG. The following cases proves uniformity among each degree.

**Case (i).** Uniformity of effective and strong degree: On considering the super subdivided vertices $w_{(p-1)t}$, all edges connected to this vertex are strong. By Definition 3.1, each edge of the FSSG satisfies the strong edge condition i.e. $\mu_{SS}(v_i, w_{(p-1)t}) > 0$ and $\mu_{SS}(v_i, w_{(p-1)t}) \geq CONN_{SS_f(G)-(v_i, w_{(p-1)t})}(v_i, w_{(p-1)t})$ . Therefore, the sum of membership values of edges incident at $w_{(p-1)t}$ will be equal to the sum of membership values of strong edges incident at $w_{(p-1)t}$. Hence, $d_{E(SS)}(w_{(p-1)t}) = d_{S(SS)}(w_{(p-1)t})$.

**Case (ii).** Uniformity of effective and strong degree: Since all the edges of FSSG are strong, effective degree includes all the membership values of strong edges. This equals to the sum of the membership value of adjacent vertices of $w_{(p-1)t}$. Therefore, $d_{E(SS)}(w_{(p-1)t}) = d_{N(SS)}(w_{(p-1)t})$.

**Case (iii).** Uniformity of strong and neighborhood degree: from the above uniformity, we get $d_{E(SS)}(w_{(p-1)t}) = d_{S(SS)}(w_{(p-1)t}) = d_{N(SS)}(w_{(p-1)t})$.

**Proposition 5.10.** In a fuzzy super subdivided graph, $d_{N(SS)}(v_i) = d_{EN(SS)}(v_i) = d_{SN(SS)}(v_i)$.

**Proof** Consider an arbitrary vertex $v_i \in V_{SS}$ of $SS_f(G)$. Let $w_{11}, w_{12}, \ldots, w_{1t}$ be the vertices adjacent to the vertex $v_i$, then by Definition 5.2, $d_{N(SS)}(v_i) = \sum_{\forall v_j \text{ adjacent to } v_i} \sigma_{SS}(v_j)$. Given that $SS_f(G)$ is a complete fuzzy super subdivided graph, every vertex is connected to all other vertices via effective and strong edges. Therefore, neighborhood degree of a vertex includes, effective and strong edges adjacent to the vertex $v_i$, i.e.,

$$d_{N(SS)}(v_i) = \sum_{\forall v_j \in E(SS) \text{ adjacent to } v_i} \sigma_{SS}(v_j) \quad \text{- (5.3)}$$

$$d_{N(SS)}(v_i) = \sum_{\forall v_j \in S(SS) \text{ adjacent to } v_i} \sigma_{SS}(v_j) \quad \text{- (5.4)}$$

By Theorem 5.6, $SS_f(G)$ is a complete fuzzy super subdivided graph, every vertex has the same effective degree as its strong degree. From (5.3) and (5.4),

$$\sum_{\forall v_j \in E(SS) \text{ adjacent to } v_i} \sigma_{SS}(v_j) = \sum_{\forall v_j \in S(SS) \text{ adjacent to } v_i} \sigma_{SS}(v_j).$$

Therefore,

$$d_{N(SS)}(v_i) = d_{EN(SS)}(v_i) = d_{SN(SS)}(v_i)$$

**6. Application**

Ecosystems are dynamic networks in which species interact with each other and environmental factors. To maintain the complex network of biodiversity and the health of the environment, this interaction contributes to the stability and resilience of the ecosystem [4]. A fuzzy super subdivision graph is a specialized graph that can model gradual transitions and overlapping relations observed in the function of the ecosystem. In Figure 9, we have considered a basic food web model for this case study.

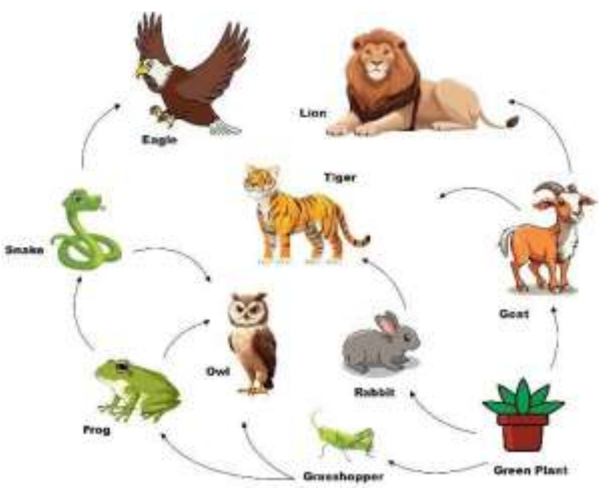


**Figure 9.** Food web representation of ecosystem

In this model, the species of the food web are pictured as vertices of the fuzzy graph. The feeding interaction with various factors between each species is represented as the edges of the fuzzy graph. The interactions between the species, such as predation, competition, and symbiosis are modeled as fuzzy super subdivided vertices. The membership values of the vertices indicate the importance of the species in the food web. Based on the strength and priority of the interactions and the intensity of the factors between each species, the membership value of fuzzy super subdivided edges is given Figure 10.

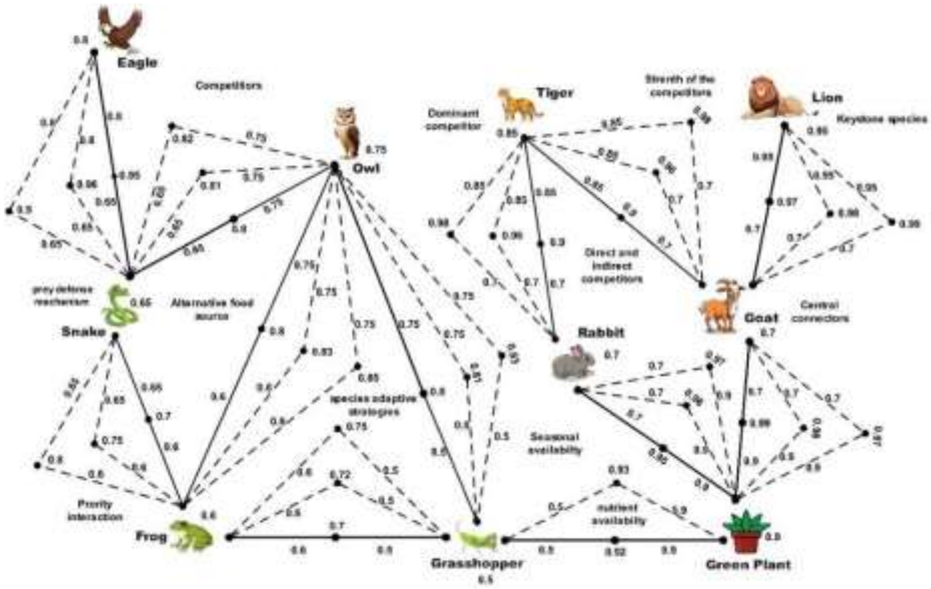
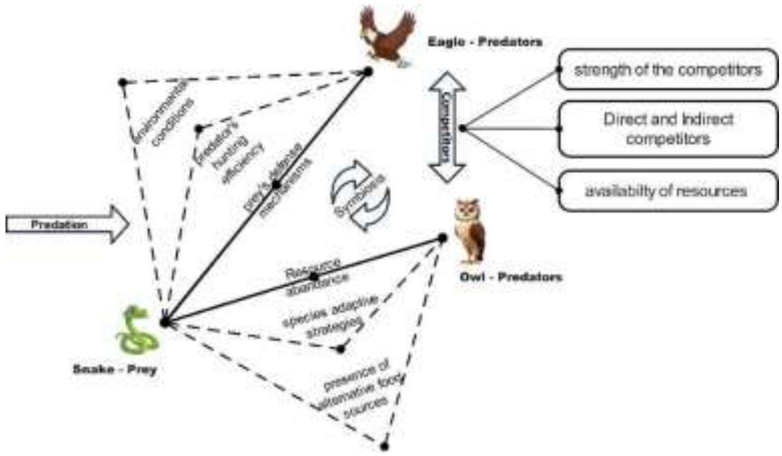



**Figure 10.** Fuzzy super subdivision model of food web

**Strength and type of interaction factors:** Interactions such as predation, competition, symbiosis that influence the relationship between prey and the predator can be analysed and understood depending on $\alpha$, $\beta$, and ð type of edges. Fuzzy super subdivision model represents different layers of interaction strength within the food web. Figure 11 represents the factors influencing the strength of interactions in fuzzy super subdivision food web.

In a food chain involving: plants → grasshoppers → frogs → snakes → Eagle/Owl, each link can be super subdivided to reflect varying factors of interaction under different environmental conditions, seasons, or resource availability.



International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 5, 2025
https://theaspd.com/index.php

**Figure 11.** Factors influencing the strength of interactions in fuzzy super subdivision food web

**Case(i).** Consider the interaction between the lion and the goat. When a minor disturbance (e.g., drought or fire) occurs, the lion can still rely on the goat for food source. The lion's population may not be significantly affected as it can hunt goats in various conditions. Hence, the lion-goat interaction remains α-strong with high dependency than the disruptions.

**Case(ii).** The interaction between the owl and snake is considered. For instance, in the case of disease outbreak which reduces the snake population, the owl needs to adjust its diet and prey on alternative species, possibly frogs or grasshoppers. The owl's population may decline if the alternative prey is scarce, but the system remains adaptable. The interaction strength weakens depends on the adverse conditions is likely a $\beta$-strong interaction. This signifies that the owl-snake interaction is important but not critical to the ecosystem's overall stability.

**Case(iii).** The eagle and the frog interaction are considered. Eagles primarily hunt other animals like snakes, rabbits, or small birds. However, in rare instances, they may prey on frogs if there is an abundance of frogs and a lack of their usual prey. This is an indirect and opportunistic interaction, not crucial for either species survival. This reflects a very weak dependency and rare ð interaction strength, for the overall stability of the ecosystem.

The range of the membership value of each species under the above-mentioned interaction types are given in Table 3.

| Interaction Type | Strength of the Interaction | Range of the vertex membership value |
|---|---|---|
| $\alpha$- strong interaction | Robust and stable despite disturbances. | $0.95 - 0.8$ |
| $\beta$- strong interaction | Moderately sensitive to environmental shifts | $0.75 - 0.5$ |
| ð- arc interaction | Rare, indirect interaction in specific circumstances. | $0.45 - 0.1$ |

**Table 3**. Type of interactions with their range of membership value

**Degrees of species:** The degrees of each species represent the ecological significance indicating their role as keystone species, dominant competitors and central connectors.

**(i) Keystone species:** From the Figure 10, the lion is a keystone species as it regulates prey population (e.g., goats) to avoid overgrazing and to maintain ecosystem balance. In the fuzzy super subdivision model (Figure 11), the top predator lion highlights strong predation interaction which is crucial for ecosystem's stability.

**(ii) Dominant competitors:** Tigers compete with lions for similar prey which includes goats and other herbivores. In this model, the tiger represents the strong competitive interactions for resources, making it a dominant competitor. Tigers exhibit a high neighborhood degree due to their multiple interactions, though their centrality is moderate.

**(iii) Central connectors:** In this context, goats are the central connectors as they connect autotrophs with herbivores. With multiple moderate membership connections, goats play a major role in energy flow, reflected in their high effective neighborhood degree. On comparing the degree values of keystone species, dominant competitor and central connector from the Table 4 below, we observe keystone species ¡ dominant competitor = central connector. The degree of each species interaction and their total degree can be formulated using:

$$D_{e(SS)}(species\ interaction) = d_{SS}(species\ 1) + d_{SS}(species\ 2) - 2\,(\sum_{\sigma} interaction\ factors)$$

| Species | Strong Degree | Neighborhood Degree | Effective Neighborhood Degree |
|---|---|---|---|
| Lion (Keystone) | 2. 85 | 2.94 | 2.94 |
| Tiger (Dominant) | 5.1 | 5.68 | 5.68 |

| | | | |
|---|---|---|---|
| Goat (Connector) | 6.3 | 8.72 | 8.72 |

**Table 4.** Degrees of Keystone species, Dominant competitors and central connectors

$$TD_{e(SS)}(species\ interaction) = D_{e(SS)}(species\ interaction) + \sum_{\sigma} interaction\ factors$$

and the calculations are given in the Table 5.

| Species (Edge) | Degree of species interaction | Total degree of species interaction |
|---|---|---|
| Lion → Goat | 3.27 | 6.21 |
| Tiger → Goat | 5.72 | 8.56 |
| Tiger → Rabbit | 3.62 | 6.46 |
| Goat → Plant | 7.62 | 10.56 |
| Rabbit → Plant | 5.64 | 8.52 |
| Grasshopper → Plant | 7.5 | 9.35 |
| Frog → Grasshopper | 3.26 | 5.43 |
| Owl → Grasshopper | 3.62 | 6.06 |
| Owl → Frog | 7.19 | 9.67 |
| Snake → Frog | 4.95 | 7.2 |
| Owl → Snake | 5.49 | 7.92 |
| Eagle → Snake | 2.63 | 5.44 |

**Table 5.** Degree and total degree of species interaction of the food web

Understanding Ecosystem Stability and Resilience: By analyzing the ecosystem responses to disturbances such as species loss or habitat fragmentation, we provide insights into the ecosystem's stability and resilience. We find stability index based on weighted degree to determine the degree of connectivity and stability level of interconnections. The weighted degree of a species is calculated by summing up all the total degree of species interaction.

$$Weighted\ degree\ of\ species = \sum(total\ degree\ of\ species\ interaction)$$

The stability index of the model is formulated based on weighted degrees of species and the number of interactions in the particular system. The higher value of the index makes the system more stable as it implies on strongly interconnected species. Conversely, the lower index indicates weak connections.

Calculating the stability index:

$$Stability\ Index = \frac{\sum(total\ degree\ of\ species\ interactions)}{maximum\ possible\ interactions}$$

Sum of the total degree of species interaction from the Table 5 is

$$\sum(total\ degree\ of\ species\ interactions) = 99.48$$

The maximum possible interactions between the given 10 species in the food web with 12 interactions is 10 × 12 = 120.

$$Stability\ Index = \frac{99.48}{120} = 0.829$$

The maximum possible stability level of the fuzzy super subdivided model of ecosystem is 82.9% with the strong interconnections. The ability of the fuzzy graph model to reconfigure itself by maintaining or restoring strong connections, redundancy and restoration and reintroduction of species or interactions after perturbation holds resilience.

**Conclusion**

This paper is the comprehensive study about structural behavior of fuzzy super subdivision graphs. In this sequel, we have established key properties and findings about structural properties. We have demonstrated that all fuzzy super subdivided edges are effective and strong, highlighting the robust nature of the graphs. One of the significant findings is that the weakest edge of all the strongest paths provides the $\beta$ −strong edges, which determines the connectivity of edge strength. The study further establishes concepts on fuzzy super subdivision subgraphs including induced, spanning, vertex-induced, edge-induced, and maximum spanning subgraphs. By analyzing and comparing these subgraphs we gained valuable insights on the structural properties of FSSG. We found that every fuzzy super subdivided induced subgraph is a fuzzy super subdivided subgraph but the converse is not true. Additionally, any subgraph consists of one point union of $m$ pendent edges, in any vertex-induced subgraph of a fuzzy super subdivided graph. A significant part of this research involves comparing effective degrees, strong degrees, neighborhood degrees within FSSG. The comparisons between these degrees brings out the crucial understanding of the recurrence pertaining to connectivity. For instance, the effective degree, strong degree and neighborhood degree of the super subdivided vertices exhibits uniformity. Overall, the paper advances the field of fuzzy graph theory by introducing new definitions, theoretical insights and illustrations of FSSG. This layer of intricate comparisons and analysis provides better understanding of complexity. This work lays groundwork for future research on these findings to explore further applications and extensions of fuzzy super subdivided graphs.

**References**

[1]. A. Rosenfeld, Fuzzy Graphs, Fuzzy Sets and their Applications to Cognitive and Decision Processes, Academic Press, (1975) 77–95. https://doi.org/10.1016/B978-0-12-775260-0.50008-6
[2]. A. Bhattacharya, M. Pal, A fuzzy graph theory approach to the facility location problem: A case study in the Indian banking system, Mathematics, 11(13) (2023), 2992.
[3]. J.N. Mordeson, C.S. Peng, Operations on fuzzy graphs, Information Sciences, 79 (1994), 159–170. https://doi.org/10.1016/0020-0255(94)90116-3K.
[4]. K. Van Meerbeek, T. Jucker and J. C. Svenning, Unifying the concepts of stability and resilience in ecology, Journal of Ecology, 109(9) (2021), 3114–3132.
[5]. K.R. Bhutani and A. Rosenfeld, Fuzzy end nodes in fuzzy graphs, Information Sciences, 152 (2003), 323–326. https://doi.org/10.1016/S0020-0255(03)00078-1
[6]. K.R. Bhutani and A. Rosenfeld, Strong arcs in fuzzy graphs, Information Sciences, 152 (2003), 319–322. https://doi.org/10.1016/S0020-0255(02)00411-5
[7]. L.A. Zadeh, Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers, Advances in Fuzzy Systems — Applications and Theory, World Scientific, Singapore, 6 (1996).
[8]. M. Pal, S. Samanta and G. Ghorai, Modern trends in fuzzy, Berlin: Springer, (2020). https://doi.org/10.1007/978-981-15-8803-7.
[9]. M.J. Sherlin, S. Noeiaghdam, L. Savarimuthu, and B. Thangavelu, Fuzzy super subdivision model with an application in infection growth analysis, Communications of Korean Mathematical Society, 39(3) (2024), 803–819. https://doi.org/10.4134/CKMS.c230218
[10]. M.S. Sunitha and S. Mathew, Fuzzy Graph Theory: A Survey, Annals Pure Applied Mathematics, 4(1) (2013), 92-110.
[11]. O.T. Manjusha, Global domination in fuzzy graphs using strong arcs, Journal of Fuzzy Extension and Applications, 4(1) (2023), 8–17. https://doi.org/10.22105/jfea.2022.354342.1227
[12]. S.K.R. Islam, M. Pal, Hyper-connectivity index for fuzzy graph with application, TWMS Journal of Applied and Engineering Mathematics, 13 (2023), 920–936.
[13]. S. Mathew, M.S. Sunitha, Types of arcs in a fuzzy graph, Information Sciences, 179(11) (2009), 1760–1768. https://doi.org/10.1016/j.ins.2009.01.003
[14]. T. Bharathi, S. Leo, A Study on Plithogenic Product Fuzzy Graphs with Application in Social Network, Indian Journal of Science and Technology, 14(40) (2021), 3051–3063. https://doi.org/10.17485/IJST/v14i40.1298