International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 18s, 2025
https://www.theaspd.com/ijes.php

The Chromatic Number of Fuzzy Path and Its Related Graphs

M.G. Karunambigai[1], Jessalet Ann Mathew[2*]

[1,2*]Department of Mathematics, Sri Vasavi College, Erode-638 316, Tamilnadu, India.
Email: [1]karunsvc@yahoo.in

***Corresponding author :** Jessalet Ann Mathew.
Email: jessaletannmathew@gmail.com

Abstract
In this paper, the chromatic number of the middle graph, splitting graph, shadow graph, line graph, total graph, and the subdivision graph of the fuzzy path P_n is determined by using fuzzy colors based on the strength of an edge incident on a vertex. Several important properties related to the fuzzy coloring of these graphs are established. Furthermore, an application of fuzzy coloring of shadow graph of fuzzy path is given.

Keywords: *Chromatic number, Fuzzy path, Middle graph, Splitting graph, Shadow graph, Line graph, Total graph, Subdivision graph.*

1. INTRODUCTION

Graph coloring is an early and fascinating concept in graph theory. It plays a crucial role in resource allocation and task scheduling, ensuring conflict-free schedules and optimal resource utilization [1]. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Fuzzy coloring [2] is an assignment of basic or fuzzy colors to the vertices of G, and it is a proper coloring,

(i) if two vertices are connected by a *strong* edge, then they either have different basic or fuzzy colors (if necessary), or one vertex can have a basic color and the other can have a fuzzy color corresponding to different basic color.

(ii) if two vertices are connected by a *weak* edge, then they either have same or different fuzzy colors, or one vertex can have a basic color and other can have a fuzzy color corresponding to the same basic color.

The minimum number of colors (basic or fuzzy) needed for a proper fuzzy coloring of G is called the chromatic number of G, is denoted by $\chi_f(G)$. In 2005, Susana Munoz et al.[3] introduced the coloring of fuzzy graphs and also proposed a method for coloring the vertices of fuzzy graphs with a crisp vertex set and a fuzzy edge set (the type 1 fuzzy graphs). In 2006, Eslahchi and Onagh [4] introduced a similar graph coloring technique for type-2 fuzzy graphs, characterized by fuzzy vertex and fuzzy edge sets, based on the concept of strong adjacency between vertices. In 2015, a new concept of fuzzy coloring of fuzzy graphs is proposed by Sovan Samanta et al. [5], using fuzzy colors based on the strength of an edge incident to a vertex.

Furthermore, in 2024, we found the chromatic number of certain families of fuzzy graphs, such as path, cycle, star, wheel, and complete graphs, using fuzzy colors based on the strength of an edge incident to a vertex and also derived some properties on fuzzy coloring [2]. In this paper, we extend our research to determine the chromatic number of the middle graph, splitting graph, shadow graph, line graph, total graph and the subdivision graph of the fuzzy path P_n, using fuzzy colors based on the strength of an edge incident to a vertex.

This article is organized as follows: Section 1 provides an overview of fuzzy coloring of a fuzzy graph. Some fundamental concepts in fuzzy graph theory that aid in the research have been reviewed in Section 2. In Section 3, we determine the chromatic number of the middle graph, splitting graph, shadow graph, line graph, total graph and the subdivision graph of the fuzzy path P_n. In Section 4, an application of fuzzy coloring of shadow graph of a fuzzy graph is given. Section 5 presents the final conclusions of this study.

Glossary of Symbols Used

Symbol	Meaning
$G = (V, E)$	Crisp graph
$G = (V, \sigma, \mu)$	Fuzzy graph
P_n	Fuzzy path
C_n	Fuzzy cycle
K_n	Complete fuzzy graph
$M(G)$	Middle graph of a crisp graph G
$M_f(G)$	Middle graph of a fuzzy graph G
$S(G)$	Splitting graph of a crisp graph G
$S_f(G)$	Splitting graph of a fuzzy graph G
$D_2(G)$	Shadow graph of a crisp graph G
$D_{2_f}(G)$	Shadow graph of a fuzzy graph G
$L(G)$	Line graph of a crisp graph G
$L_f(G)$	Line graph of a fuzzy graph G
$T(G)$	Total graph of a crisp graph G
$T_f(G)$	Total graph of a fuzzy graph G
$sd(G)$	Subdivision graph of a crisp graph G
$sd_f(G)$	Subdivision graph of a fuzzy graph G

2. PRELIMINARIES

The definitions from the fuzzy graph theory and the fuzzy coloring, which aid in determining the chromatic number of various fuzzy graphs, are reviewed in this section.

Definition 2.1.[6] A fuzzy graph $G = (V, \sigma, \mu)$ is a pair of functions (σ, μ), where $\sigma : V \rightarrow [0, 1]$ is a fuzzy subset of a non-empty set V, and $\mu : V \rightarrow [0, 1]$ is a symmetric fuzzy relation on σ, such that the relation $\mu(v_i, v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$ is satisfied for all $v_i, v_j \in V$ and $(v_i, v_j) \in E \subset V \times V$. Here, $\sigma(v_i)$ denote the degree of membership of the vertex v_i, and $\mu(v_i, v_j)$ denotes the degree of membership of the edge relation $e_{ij} = (v_i, v_j)$ on $V \times V$.

Note : In this paper, we denote $\sigma(v_i) \wedge \sigma(v_j) = min\{\sigma(v_i), \sigma(v_j)\}$ and $\sigma(v_i) \vee \sigma(v_j) = max\{\sigma(v_i), \sigma(v_j)\}$.

Definition 2.2.[7] Let $G = (V, \sigma, \mu)$ be a fuzzy graph with underlying crisp graph G^*. A fuzzy path P_n in G is a sequence of distinct vertices $v_0, v_1, \dots, v_n$ such that $\mu(v_{i-1}, v_i) > 0, 1 \leq i \leq n$. Here $n \geq 1$ is called the length of the path P_n.

Definition 2.3. [7] A fuzzy path P_n in which $v_0 = v_n$ and $n \geq 3$, then P_n is called a fuzzy cycle C_n of length n.

Definition 2.4. [3] Let $G = (V, \sigma, \mu)$ be a fuzzy graph and an edge $e = (v_i, v_j) \in G$ is called **strong** if $\frac{1}{2}\{\sigma(v_i) \wedge \sigma(v_j)\} \leq \mu(v_i, v_j)$ and it is called **weak** otherwise.

Definition 2.5.[3] Let $G = (V, \sigma, \mu)$ be a fuzzy graph and the **strength of an edge** $(v_i, v_j) \in G$ is denoted by,

$$I(v_i, v_j) = \frac{\mu(v_i, v_j)}{\sigma(v_i) \wedge \sigma(v_j)}.$$

Definition 2.6.[8] A fuzzy graph $G = (V, \sigma, \mu)$ is called a strong fuzzy graph if each edge in G is a strong edge.

Definition 2.7.[2] Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Fuzzy coloring is an assignment of basic or fuzzy colors to the vertices of a fuzzy graph G and it is proper,

(i) if two vertices are connected by a strong edge, then they either have different basic or fuzzy colors (if

necessary), or one vertex can have a basic color and the other can have a fuzzy color corresponding to different basic color.

(ii) if two vertices are connected by a weak edge, then they either have same or different fuzzy colors, or one vertex can have a basic color and other can have a fuzzy color corresponding to the same basic color.

Definition 2.8.[2] Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Perfect fuzzy coloring (optimal fuzzy coloring) is an assignment of minimum number of colors (basic or fuzzy) for a proper fuzzy coloring of G.

Definition 2.9.[2] Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The minimum number of colors (basic or fuzzy) needed for a proper fuzzy coloring of G is called the chromatic number of G and is denoted by $\chi_f(G)$.

Lemma 2.1.[2] *Let P_n be a fuzzy path of length n. If all edges are weak in P_n, then $\chi_f(P_n) = 1$.*

Lemma 2.2.[2] *Let P_n be a fuzzy path of length n. If all the edges are strong in P_n, then* $\chi_f(P_n)$= 2.

Theorem 2.1.[2] *Let P_n be a fuzzy path of length n. If atleast one edge is strong in P_n, then* $\chi_f(P_n) = 2$.

Lemma 2.3.[2] *Let C_n be a fuzzy cycle of length n. If all edges are weak in C_n, then $\chi_f(C_n) = 1$.*

Lemma 2.4.[2] *Let C_n be a fuzzy cycle of length n. If all the edges are strong in C_n, then*

$$\chi_f(C_n) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Theorem 2.2.[2] *Let C_n be a fuzzy cycle of length n. If weak and strong edges are distributed in any sequence in C_n, then*

$$\chi_f(C_n) = \begin{cases} 3 & if\ \frac{n}{2}\ number\ of\ strong\ and\ weak\ edges\ are\ alternatively \\ & distributed\ in\ C_n, where\ n(\geq 6) is\ even, \\ 2 & otherwise. \end{cases}$$

Theorem 2.3. [9] *Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two fuzzy graphs, the chromatic numbers of G_1 and G_2 be $\chi_f(G_1)$ and $\chi_f(G_2)$, respectively. If fuzzy graph $G(V, E)$ is the union of two fuzzy graphs G_1 and G_2, then the chromatic number of G satisfies $max\{\chi_f(G_1), \chi_f(G_2)\} \leq \chi_f(G) \leq \chi_f(G_1) + \chi_f(G_2)$.*

Theorem 2.4. [10] *The complete graph K_n has Hamiltonian decomposition for all n. i.e., $K_{2n+1} = \oplus\, nC_{2n+1}$ and $K_{2n} = C_{2n} \oplus nP_1$.*

Proof. The result is trivially true for $n = 1$ and $n = 2$. Let $n = 2m + 1 \geq 3$ be odd. Let the vertices of K_n be labeled $v_0, v_1, \dots, v_{2m}$. Let C be the Hamilton cycle $v_0\ v_1\ v_2\ v_{2m}\ v_3\ v_{2m-1}\ v_4\ v_{2m-2} \cdots v_{m+3}\ v_m\ v_{m+2}\ v_{m+1}\ v_0$ and let σ be the permutation $(v_0)(v_1\ v_2\ v_3 \dots v_{2m-1}\ v_{2m})$. Then $C, \sigma(C), \sigma^2(C), \sigma^{m-1}(C)$ is a Hamilton decomposition of K_n. When $n = 2m \geq 4$ is even, let the vertices of K_n be labeled $v_0, v_1, v_2, \dots, v_{2m-1}$. Let C be the Hamilton cycle $v_0\ v_1\ v_2\ v_{2m-1}\ v_3\ v_{2m-2} \dots v_{m-1}\ v_{m+2}\ v_m\ v_{m+1} v_0$ and σ be the permutation $(v_0)(v_1\ v_2\ v_3 \dots v_{2m-2}\ v_{2m-1})$. Then $C, \sigma(C), \sigma^2(C), \sigma^{m-2}(C)$ are $m - 1$ edge disjoint Hamilton cycles. The remaining edges $v_0\ v_m, v_{m-1}\ v_{m+1}, v_{m-2}\ v_{m+2}, \dots, v_1\ v_{2m-1}$ form a perfect matching.

Note [2] : i.e., $K_{2n+1} = \oplus\, nC_{2n+1}$ and $K_{2n} = C_{2n} \oplus nP_1$, where $\oplus$ denotes edge disjoint union.

3. The Chromatic Number of Some Related Graphs of Fuzzy Path

In this section, we will find the chromatic number of the middle graph $M_f(P_n)$, splitting graph $S_f(P_n)$, shadow graph $D_{2_f}(P_n)$, line graph $L_f(P_n)$, total graph $T_f(P_n)$ and the subdivision graph $sd_f(P_n)$ of the fuzzy path P_n.

Theorem 3.1. $\chi_f(G) \geq max\ \{\chi_f(G_i) : 1 \leq i \leq k\}$, *where $G = G_1 \cup G_2 \cup \dots \cup G_k$ and $G_i, 1 \leq i \leq k$ are fuzzy graphs.*

Proof. Proof follows from Theorem 2.3.

Corollary 3.1.1. $\chi_f(G) \geq max\ \{\chi_f(G_i) : 1 \leq i \leq k\}$, *where $G = G_1 \oplus G_2 \oplus \dots \oplus G_k$ and $G_i, 1 \leq i \leq k$ are edge disjoint fuzzy graphs.*

International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 18s, 2025
https://www.theaspd.com/ijes.php

3.1. The Chromatic Number of $M_f(P_n)$

Definition 3.1. The middle graph $M_f(G)(V_M, \sigma_M, \mu_M)$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $M(G)(V_M, E_M)$, with the vertex set $V_M = V \cup V_{ij}$ where $V = \{v_i \mid v_i \in V\}$ and $V_{ij} = \{v_{ij} \, \forall \, (v_i, v_j) \in E\}$ and $\nu(M_f(G)) = n + 1 + n = 2n + 1$ and the edge set

$$E_M = \begin{cases} (v_{ij}, v_i), (v_{ij}, v_j) & \forall \, i \text{ and } j, \\ (v_{ij}, v_{rs}) & \text{if the edges } (v_i, v_j) \text{ and } (v_r, v_s) \text{ are adjacent in } G. \end{cases}$$

Then, $\sigma_M(v_i) = \sigma(v_i)$ *if* $v_i \in V, 0 \leq i \leq n$,
$\sigma_M(v_{ij}) = \mu(v_i, v_j)$ *if* $(v_i, v_j) \in E \, \forall \, i$ *and* j,
$\mu_M(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G,
and $\mu_M(v_i, v_{ij}) = \mu_M(v_j, v_{ij}) = \mu(v_i, v_j) \, \forall \, i$ *and* j.

Example 1. The middle graph of P_3 is given in Figure 1.

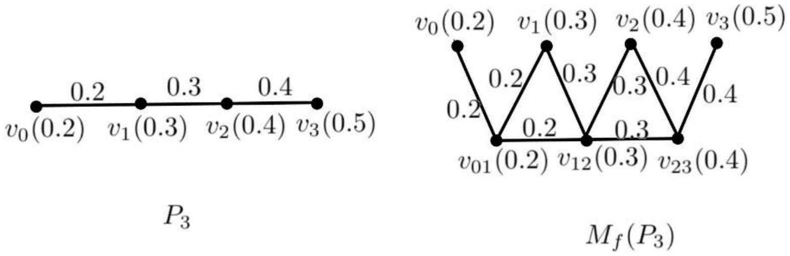

Figure 1. Fuzzy path P_3 and its middle graph $M_f(P_3)$.

Remark 3.1.1. Let $P_n: v_0 v_1 \dots v_{n-1} v_n$ be a fuzzy path of length n. Then $M_f(P_n) = \oplus \, (n-1)C_3 \oplus 2P_1$ (by Theorem 2.4), where C_3 is oriented as, $C_3 : v_{ij} \, v_{i+1j+1} \, v_{i+1} \, v_{ij}, 0 \leq i \leq n-2, 1 \leq j \leq n$ and P_1's are oriented as, $P_1: v_0 \, v_{01}$ & $P_1 : v_{n-1n} \, v_n$.

Lemma 3.1.1. *Let P_n be a fuzzy path of length n. Then $M_f(P_n)$ is a strong fuzzy graph.*

Proof. Let $P_n : v_0 v_1 \dots v_{n-1} v_n$ be a fuzzy path of length n. By the definition of middle graph of a fuzzy graph, we have
$\sigma_M(v_i) = \sigma(v_i)$*if* $v_i \in V, 1 \leq i \leq n$,
$\sigma_M(v_{ij}) = \mu(v_i, v_j)$*if* $(v_i, v_j) \in E \, \forall \, i$ & j,
$\mu_M(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G,
and $\mu_M(v_i, v_{ij}) = \mu_M(v_j, v_{ij}) = \mu(v_i, v_j) \, \forall \, i$ & j.
Then each edges of $M_f(P_n)$ satisfies the condition of a strong edge (by definition 2.4).
Therefore, $M_f(P_n)$ is a strong fuzzy graph.

Theorem 3.1.1. *If $M_f(P_n)$ is a strong fuzzy graph, then $\chi_f(M_f(P_n)) = 3$.*

Proof. Let $P_n : v_0 \, v_1 \dots v_{n-1} \, v_n$ be a fuzzy path of length n. Then $M_f(P_n) = \oplus \, (n-1)C_3 \oplus 2P_1$ (by Remark 3.1.1) and by Lemma 3.1.1}, $M_f(P_n)$ is a strong fuzzy graph. Then by Lemma 2.4 we have, $\chi_f(C_3) = 3$ and by Lemma 2.2 we have, $\chi_f(P_1) = 2$. Therefore by Corollary 3.1.1,

$$\begin{aligned} \chi_f(M_f(P_n)) &= max\{\chi_f(C_3), \chi_f(P_1)\} \\ &= \max\{3,2\} \\ &= 3. \end{aligned}$$

International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 18s, 2025
https://www.theaspd.com/ijes.php

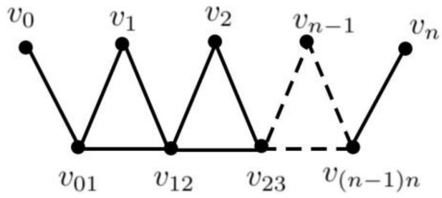

Figure 2. Middle graph of the fuzzy path P_n.

3.2. The Chromatic Number of $S_f(P_n)$

Definition 3.2. The splitting graph $S_f(G)(V_S, \sigma_S, \mu_S)$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $S(G)(V_S, E_S)$, with the vertex set $V_S = V \cup V'$ where $V = \{v_i \mid v_i \in V\}$ and $V' = \{v_i' \ \forall\, v_i \in V\}$ and the edge set

$$E_S = \begin{cases} (v_i, v_j) & if\, v_i \text{ and } v_j \text{ are adjacent in } G, \\ (v'_i, v_j) & if\, v_i' \in V' \text{and } v_j \in V \text{ that are adjacent to } v_i \in V. \end{cases}$$

Then, $\sigma_S(v_i) = \sigma_S(v_i') = \sigma(v_i)\ for\ v_i \in V\ and\ v_i' \in V'$,
$\mu_S(v_i, v_j) = \mu(v_i, v_j)\ if\ v_i\ and\ v_j\ are\ adjacent\ in\ V$,
and $\mu_S(v_i', v_j) = \sigma_S(v_i') \wedge \sigma_S(v_j)\ if\ v_i' \in V' and\ v_j \in V\ that\ are\ adjacent\ to\ v_i \in V$.

Example 2. The splitting graph of P_3 is given in Figure 3.

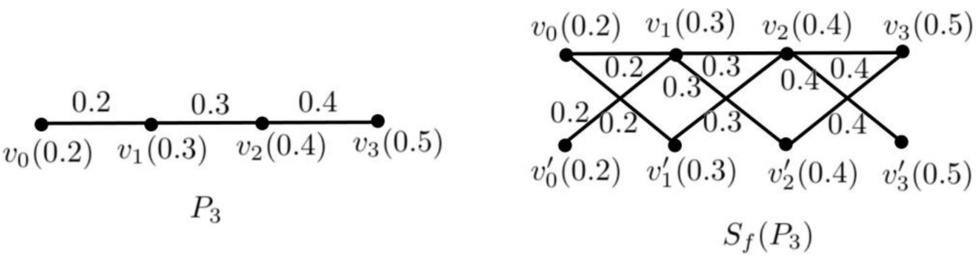

Figure 3. Fuzzy path P_3 and its splitting graph $S_f(P_3)$.

Remark 3.2.1. Let $P_n: v_0 v_1 \dots v_{n-1} v_n$ be a fuzzy path of length n.
Case 1 : In P_n, if n is odd.
Then $S_f(P_{2n+1}) = \oplus\, nC_4 \oplus P_{2n+1} \oplus P_2, n \geq 1$ (by Theorem 2.4), where C_4 is oriented as $C_4: v_i\ v_{i+1}\ v_{i+2}\ v_{i+1}'\ v_i$, for $i = 0, 2, 4, \dots n-3$, P_{2n+1} is oriented as $P_{2n+1}: v_0'\ v_1\ v_2'\ v_3 \dots v_{n-1}' v_n$ and P_2 is oriented as $P_2: v_n v_{n-1} v_n'$.

Case 2 : In P_n, if n is even.
Then $S_f(P_{2n}) = \oplus\, nC_4 \oplus P_{2n}, n \geq 1$ (by Theorem 2.4), where C_4 is oriented as $C_4: v_i\ v_{i+1}\ v_{i+2}\ v_{i+1}'\ v_i$, for $i = 0, 2, 4, \dots n-2$ and P_{2n} is oriented as $P_{2n}: v_0'\ v_1 v_2' v_3 \dots v_{n-1} v_n'$.

Lemma 3.2.1. *Let $P_n: v_0 v_1 \dots v_{n-1} v_n$ be a fuzzy path of length n. If all the edges are weak in P_n, then the edges $(v_i, v_{i+1}), 0 \leq i \leq n-1$ are weak in $S_f(P_n)$ and the edges $(v_i', v_{i+1}), 0 \leq i \leq n-1$ and $(v_i', v_{i-1}), 1 \leq i \leq n$ are strong in $S_f(P_n)$.*

Proof. Proof follows from the definition of splitting graph of a fuzzy graph and the definition of weak and strong edges.

Theorem 3.2.1. *Let P_n be a fuzzy path of length n. If all the edges are weak in P_n, then $\chi_f(S_f(P_n)) = 2$.*
Proof. Let $P_n: v_0 v_1 \dots v_{n-1} v_n$ be a fuzzy path of length n. Then by Lemma 3.2.1, the edges $(v_i, v_{i+1}), 0 \le i \le n-1$ are weak in $S_f(P_n)$and the edges $(v_i', v_{i+1}), 0 \le i \le n-1$ and $(v_i', v_{i-1}), 1 \le i \le n$ are strong in $S_f(P_n)$.
Case 1 : In P_n, if n is even.
Then $S_f(P_{2n}) = \oplus\, nC_4 \oplus P_{2n}, n \ge 1$ (by Remark 3.2.1). Then by Theorem 2.2 we have, $\chi_f(C_4) = 2$ and by Lemma 2.2 we have, $\chi_f(P_{2n}) = 2$. Therefore by Corollary 3.1.1, $\chi_f(S_f(P_{2n})) = 2$.

Case 2 : In P_n, if n is odd.
Then $S_f(P_{2n+1}) = \oplus\, nC_4 \oplus P_{2n+1} \oplus P_2, n \ge 1$ (by Remark 3.2.1). Then by Theorem 2.2 we have, $\chi_f(C_4) = 2$, by Lemma 2.2 we have, $\chi_f(P_{2n+1}) = 2$ and by Theorem 2.1 we have, $\chi_f(P_2) = 2$.
Therefore by Corollary 3.1.1, $\chi_f(S_f(P_{2n+1})) = 2$.

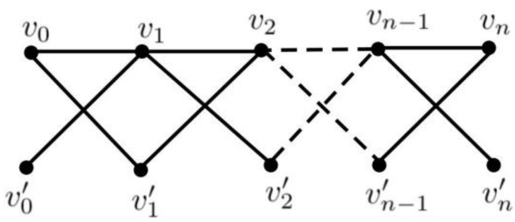

Figure 4. Splitting graph of the fuzzy path P_n .

Note : In P_n, if $n = 1, S_f(P_1) = P_3$. $\therefore \chi_f(S_f(P_1)) = 2$.

Lemma 3.2.2. *Let P_n be a fuzzy path of length n. If all the edges are strong in P_n, then $S_f(P_n)$ is a strong fuzzy graph.*
Proof. Proof follows from the definition of splitting graph of a fuzzy graph and the definition of strong edge.

Theorem 3.2.2. *If $S_f(P_n)$ is a strong fuzzy graph, then* $\chi_f(S_f(P_n)) = 2$. (The proof will be similar as above theorem).

Lemma 3.2.3. *Let $P_n: v_0 v_1 \dots v_{n-1} v_n$ be a fuzzy path of length n. If weak and strong edges are distributed in any sequence in P_n, then the edges $(v_i, v_{i+1}), 0 \le i \le n-1$ are also weak and strong, which are distributed in any sequence in $S_f(P_n)$ while the edges $(v_i', v_{i+1}), 0 \le i \le n-1$ and $(v_i', v_{i-1}), 1 \le i \le n$ are strong in $S_f(P_n)$.*
(The proof will be similar as above lemma).

Theorem 3.2.3. *Let P_n be a fuzzy path of length n. If weak and strong edges are distributed in any sequence in P_n, then $\chi_f(S_f(P_n)) = 2$.* (The proof will be similar as above theorem).

3.3. The Chromatic Number of $D_{2_f}(P_n)$

Definition 3.3. The shadow graph $D_{2_f}(G)(V_{D_2}, \sigma_{D_2}, \mu_{D_2})$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $D_2(G)(V_{D_2}, E_{D_2})$ is obtained by taking two copies of G namely G' and G'' with the vertex set $V_{D_2} = V' \cup V''$ where $V' = \{v_i' \ \forall\, v_i \in V\}$ and $V'' = \{v_i'' \ \forall\, v_i \in V\}$ and the edge set

$$E_{D_2} = \begin{cases} (v_i', v_j'), (v_i'', v_j'') & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G, \\ (v_i', v_j'') & \text{if } v_i' \in V' \text{and } v_j'' \in V'' \text{that are adjacent to } v_i'' \in V''. \end{cases}$$

Then, $\sigma_{D_2}(v') = \sigma_{D_2}(v'') = \sigma(v) for\ v \in V, v' \in V', v'' \in V''$,
$\mu_{D_2}(v_i' v_j') = \mu_{D_2}(v_i'' v_j'') = \mu(v_i v_j)$, for $v_i', v_j' \in V', v_i'', v_j'' \in V'', v_i, v_j \in V$,
$\mu_{D_2}(v_i' v_j'') = \sigma(v_i') \wedge \sigma(v_j'')\ if\ v_i' \in V' and\ v_j'' \in V''$ that are adjacent to $v_i'' \in V''$.

International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 18s, 2025
https://www.theaspd.com/ijes.php

Example 3. The shadow graph of P_3 is given in Figure 5.

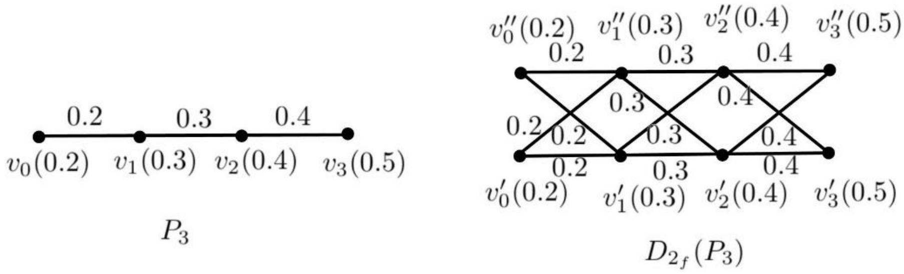

Figure 5. Fuzzy path P_3 and its shadow graph $D_{2_f}(P_3)$.

Remark 3.3.1. Let $P_n: v_0 v_1 \dots v_{n-1} v_n$ be a fuzzy path of length n. Then $D_{2_f}(P_n) = \bigoplus nC_4$ (by Theorem 2.4), where C_4 is oriented as $C_4: v_i'\, v_{i+1}'\, v_i''\, v_{i+1}''\, v_i'$, for $i = 0, 1, \dots, n-1$.

Lemma 3.3.1. *Let $P_n: v_0\, v_1 \dots v_{n-1} v_n$ be a fuzzy path of length n. If all the edges are weak in P_n, then the edges $(v_i', v_{i+1}'), 0 \le i \le n-1$ and $(v_i'', v_{i+1}''), 0 \le i \le n-1$ are weak in $D_{2_f}(P_n)$ and the edges $(v_i', v_{i+1}''), 0 \le i \le n-1$ and $(v_i', v_{i-1}''), 1 \le i \le n$ are strong in $D_{2_f}(P_n)$.*
Proof. Proof follows from the definition of shadow graph of a fuzzy graph and the definition of weak and strong edges.

Theorem 3.3.1. *Let P_n be a fuzzy path of length n. If all the edges are weak in P_n, then $\chi_f(D_{2_f}(P_n) = 2$.*
Proof. Let $P_n: v_0\, v_1 \dots v_{n-1} v_n$ be a fuzzy path of length n. Then $D_{2_f}(P_n) = \bigoplus nC_4$ (by Remark 3.3.1) and by Lemma 3.3.1, the edges $(v_i', v_{i+1}'), 0 \le i \le n-1$ and $(v_i'', v_{i+1}''), 0 \le i \le n-1$ are weak in $D_{2_f}(P_n)$ and the edges $(v_i', v_{i+1}''), 0 \le i \le n-1$ and $(v_i', v_{i-1}''), 1 \le i \le n$ are strong in $D_{2_f}(P_n)$. Then by Theorem 2.2, we have $\chi_f(C_4) = 2$. Therefore by Corollary 3.1.1, $\chi_f(D_{2_f}(P_n)) = 2$.

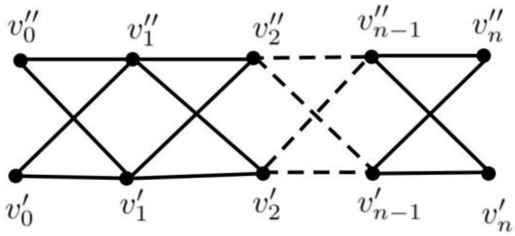

Figure 6. Shadow graph of the fuzzy path P_n.

Lemma 3.3.2. *Let P_n be a fuzzy path of length n. If all the edges are strong in P_n, then $D_{2_f}(P_n)$ is a strong fuzzy graph.*
Proof. Proof follows from the definition of shadow graph of a fuzzy graph and the definition of strong edge.

Theorem 3.3.2. *If $D_{2_f}(P_n)$ is a strong fuzzy graph, then $\chi_f(D_{2_f}(P_n)) = 2$.* (The proof will be similar as above theorem).

Lemma 3.3.3. *Let $P_n: v_0\, v_1 \dots v_{n-1}\, v_n$ be a fuzzy path of length n. If weak and strong edges are distributed in any sequence in P_n, then the edges $(v_i', v_{i+1}'), 0 \le i \le n-1$ and $(v_i'', v_{i+1}''), 0 \le i \le n-1$ are weak and strong, which are distributed in any sequence in $D_{2_f}(P_n)$ while the edges $(v_i', v_{i+1}''), 0 \le i \le n-1$ and $(v_i', v_{i-1}''), \le i \le n$ are strong in $D_{2_f}(P_n)$.* (The proof will be similar as above lemma).

Theorem 3.3.3. *Let P_n be a fuzzy path of length n. If weak and strong edges are distributed in any sequence in P_n, then* $\chi_f(D_{2_f}(P_n)) = 2$. (The proof will be similar as above theorem).

3.4. The Chromatic Number of $L_f(P_n)$

Definition 3.4. The line graph $L_f(G)(V_L, \sigma_L, \mu_L)$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $L(G)(V_L, E_L)$, where the vertex set $V_L = \{v_{ij} \forall (v_i, v_j) \in E\}$ and edge set $E_L = \{(v_{ij}, v_{rs}) \mid \textit{the edges } (v_i, v_j) \textit{ and } (v_r, v_s) \textit{ are adjacent in } G\}$. Then, $\sigma_L(v_{ij}) = \mu(v_i, v_j)$ if $v_{ij} \in V_L$ and $\mu_L(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G.

Example 4. The line graph of P_3 is given in Figure 7.

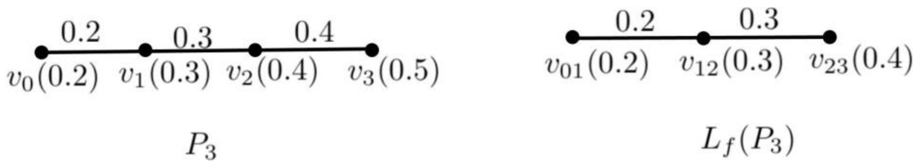

Figure 7. Fuzzy path P_3 and its line graph $L_f(P_3)$.

Lemma 3.4.1. *Let P_n be a fuzzy path of length n. Then $L_f(P_n)$is a strong fuzzy graph.*
Proof. Proof follows from the definition of line graph of a fuzzy graph and the definition of strong edge.

Theorem 3.4.1. *If $L_f(P_n)$ is a strong fuzzy graph, then* $\chi_f(L_f(P_n)) = 2$.
Proof. Let $P_n: v_0\, v_1 \dots v_{n-1} v_n$ be a fuzzy path of length n. Since $L_f(P_n) \cong P_{n-1}$, by Lemma 2.2, $\chi_f(L_f(P_n)) = 2$.

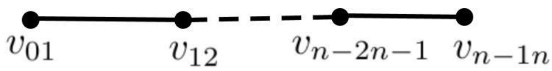

Figure 8. Line graph of the fuzzy path P_n.

3.5. The Chromatic Number of $T_f(P_n)$

Definition 3.5. The total graph $T_f(G)(V_T, \sigma_T, \mu_T)$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $T(G)(V_T, E_T)$, with the vertex set $V_T = V \cup V_{ij}$ where $V = \{v_i \mid v_i \in V\}$ and $V_{ij} = \{v_{ij} \forall (v_i, v_j) \in E\}$ and the edge set

$$E_T = \begin{cases} (v_i, v_j) & \textit{if } v_i \textit{ and } v_j \textit{ are adjacent in } G, \\ (v_{ij}, v_i), (v_{ij}, v_j) & \forall\ i \textit{ and } j, \\ (v_{ij}, v_{rs}) & \textit{if the edges } (v_i, v_j) \textit{and } (v_r, v_s) \textit{are adjacent in } G. \end{cases}$$

Then, $\sigma_T(v_i) = \sigma(v_i)\ if\ v_i \in V$,
$\sigma_T(v_{ij}) = \mu(v_i, v_j)\ if\ (v_i, v_j) \in E\ \forall\ i\ and\ j$,
$\mu_T(v_i, v_j) = \mu(v_i, v_j)\ if\ v_i\ and\ v_j\ are\ adjacent\ in\ G$,
$\mu_T(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G,
and $\mu_T(v_i, v_{ij}) = \mu_T(v_j, v_{ij}) = \mu(v_i, v_j) \forall\ i\ and\ j$.

International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 18s, 2025
https://www.theaspd.com/ijes.php

Example 5. The total graph of P_3 is given in Figure 9.

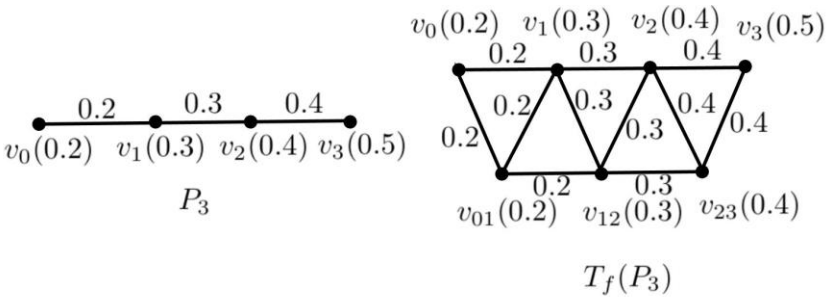

Figure 9. Fuzzy path P_3 and its total graph $T_f(P_3)$.

Remark 3.5.1. Let $P_n: v_0v_1 \dots v_{n-1}v_n$ be a fuzzy path of length n. Then $T_f(P_n) = P_n \oplus L_f(P_n) \oplus P_{2n}$ (by Theorem 2.4), where P_n is oriented as $P_n: v_0\, v_1 \dots v_{n-1}\, v_n$, $L_f(P_n)$ is oriented as $L_f(P_n): v_{01}v_{12} \dots v_{n-1n}$ and P_{2n} is oriented as $P_{2n} : v_0\, v_{01}\, v_1\, v_{12} \dots v_{n-1n}\, v_n$.

Lemma 3.5.1. *Let $P_n : v_0\, v_1 \dots v_{n-1}\, v_n$ be a fuzzy path of length n. If all the edges are weak in P_n, then the edges of $P_n \in T_f(P_n)$ are weak and the edges of $L_f(P_n) \in T_f(P_n)$ and $P_{2n} \in T_f(P_n)$ are strong.*
Proof. Proof follows from the definition of total graph of a fuzzy graph and the definition of weak and strong edges.

Theorem 3.5.1. *Let P_n be a fuzzy path of length n. If all the edges are weak in P_n, then* $\chi_f(T_f(P_n)) = 2$.
Proof. Let $P_n : v_0\, v_1 \dots v_{n-1}v_n$ be a fuzzy path of length n. Then $T_f(P_n) = P_n \oplus L_f(P_n) \oplus P_{2n}$ (by Remark 3.5.1) and by Lemma 3.5.1, then the edges of P_n are weak in $T_f(P_n)$and the edges of $L_f(P_n)$ and P_{2n} are strong in $T_f(P_n)$. Then by Lemma 2.1 we have, $\chi_f(P_n) = 2$, by Theorem 3.4.1 we have, $\chi_f(L_f(P_n)) = 2$ and by Lemma 2.2 we have, $\chi_f(P_{2n}) = 2$. Therefore by Corollary 3.1.1,

$$\begin{aligned}\chi_f\left(T_f(P_n)\right) &= \max\{\chi_f(P_n), \chi_f(L_f(P_n)), \chi_f(P_{2n})\\ &= max\{2,2,2\}\\ &= 2.\end{aligned}$$

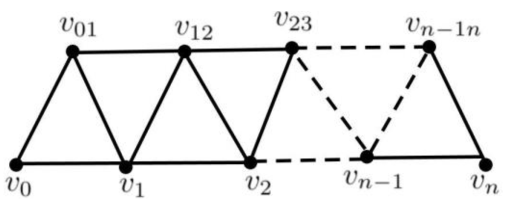

Figure 10. Total graph of the fuzzy path P_n.

Lemma 3.5.2. *Let P_n be a fuzzy path of length n. If all the edges are strong in P_n, then $T_f(P_n)$ is a strong fuzzy graph.*
Proof. Proof follows from the definition of total graph of a fuzzy graph and the definition of strong edge.

Theorem 3.5.2. *If $T_f(P_n)$ is a strong fuzzy graph, then $\chi_f(T_f(P_n)) = 3$.*
Proof. Let $P_n : v_0\, v_1 \dots v_{n-1}\, v_n$ be a fuzzy path of length n. Then $T_f(P_n) = P_n \oplus L_f(P_n) \oplus P_{2n}$ (by Remark 3.5.1) and by Lemma 3.5.2, all edges are strong in $T_f(P_n)$. Then by Theorem 3.4.1 we have, $\chi_f(L_f(P_n)) = 2$ and by Lemma 2.2 we have, $\chi_f(P_n) = \chi_f(P_{2n}) = 2$. Then by Corollary 3.1.1,

$$\chi_f(T_f(P_n)) = max\{\chi_f(P_n), \chi_f(L_f(P_n)), \chi_f(P_{2n})\}$$
$$= max\{2,2,2\} + 1$$
$$= 3.$$

Lemma 3.5.3. *Let $P_n : v_0\, v_1 \dots v_{n-1} v_n$ be a fuzzy path of length n. If weak and strong edges are distributed in any sequence in P_n, then the edges of $P_n \in T_f(P_n)$ are weak and strong, which are distributed in any sequence in $T_f(P_n)$, while the edges of $L_f(P_n) \in T_f(P_n)$ and $P_{2n} \in T_f(P_n)$ are strong.* (The proof will be similar as above lemma).

Theorem 3.5.3. *Let P_n be a fuzzy path of length n. If weak and strong edges are distributed in any sequence in P_n, then $\chi_f(T_f(P_n)) = 2$.* (The proof will be similar as above theorem).

3.6. The Chromatic Number of $sd_f(P_n)$

Definition 3.6. The subdivision graph $sd_f(G)(V_{sd}, \sigma_{sd}, \mu_{sd})$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $sd(G)(V_{sd}, E_{sd})$, with the vertex set $V_{sd} = V \cup V_{ij}$, where $V = \{v_i \mid v_i \in V\}$ and $V_{ij} = \{v_{ij} \; \forall \; (v_i, v_j) \in E\}$ and the edge set $E_{sd} = \{(v_{ij}, v_i), (v_{ij}, v_j) \forall\; i \;\&\; j\}$. Then, $\sigma_{sd}(v_{ij}) = \mu(v_i, v_j)$ if $(v_i, v_j) \in E \; \forall \; i \;\&\; j$ and $\mu_{sd}(v_i, v_{ij}) = \mu_{sd}(v_j, v_{ij}) = \mu(v_i, v_j) \; \forall \; i \;\&\; j$.

Example 6. The subdivision graph of P_3 is given in Figure 11.

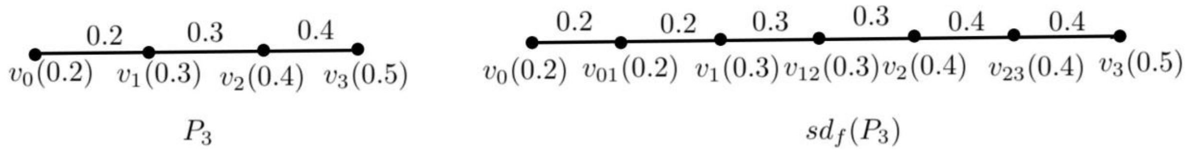

Figure 11. Fuzzy path P_3 and its subdivision graph $sd_f(P_3)$.

Lemma 3.6.1. *Let P_n be a fuzzy path of length n. Then $sd_f(P_n)$ is a strong fuzzy graph.*
Proof. Proof follows from the definition of subdivision graph of a fuzzy graph and the definition of strong edge.

Theorem 3.6.1. *If $sd_f(P_n)$ is a strong fuzzy graph, then $\chi_f(sd_f(P_n)) = 2$.*
Proof. Let $P_n : v_0\, v_1 \dots v_{n-1}\, v_n$ be a fuzzy path of length n. Since $sd_f(P_n) \cong P_{2n}$, by Lemma 2.2, $\chi_f(sd_f(P_n)) = 2$.

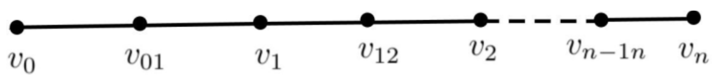

Figure 12. Subdivision graph of the fuzzy path P_n.

4. Application

Smart cities depend on interconnected hubs to manage traffic, emergencies, and safety systems. However, during emergency situations like power outages or natural disasters, maintaining reliable communication is a critical challenge. This study applies shadow graph of a fuzzy graph to model conflict-free routing in an affected hub's communication network, ensuring uninterrupted connectivity. Furthermore, it determines the minimum number of channels required for interference-free communication under such conditions.

Let G be a fuzzy graph that represents a communication network with vertices A, B, C and D. Each vertex corresponds to a smart communication hub located in a different district. The membership value of each vertex indicates the operational reliability of the hubs in each district, which are $0.6, 0.4, 0.5$ and 0.3, respectively. Two vertices are connected if and only if there exists a direct communication link between them, and the membership value of each edge reflects the reliability of the communication links. Let the edges $(A, B), (B, C), (C, D)$ represent the direct communication link between the smart hubs with membership values $0.4, 0.3$ and 0.2

respectively. Moreover, all the edges are strong in G, representing the strong communication links between the hubs.

Construct the shadow graph $D_{2_f}(G)$ of fuzzy graph G (by definition 3.3), then the edges $(A', B'), (B', C'), (C', D')$ represent the direct communication links between the smart hubs, while the edges $(A'', B''), (B'', C''), (C'', D'')$ represent the direct communication link between the backup hubs. These edges retain the same membership values as those of the corresponding edges in G. Additionally, the edges $(A', B''), (B', C''), (C', D''), (B', A''), (C', B''), (D', C'')$ represent the direct communication links between the smart hubs and backup hubs, with membership values $0.4, 0.4, 0.3, 0.4, 0.4$ and 0.3 (Figure 13). Since G is strong, $D_{2_f}(G)$ is also strong (Corollary 3.3.2).

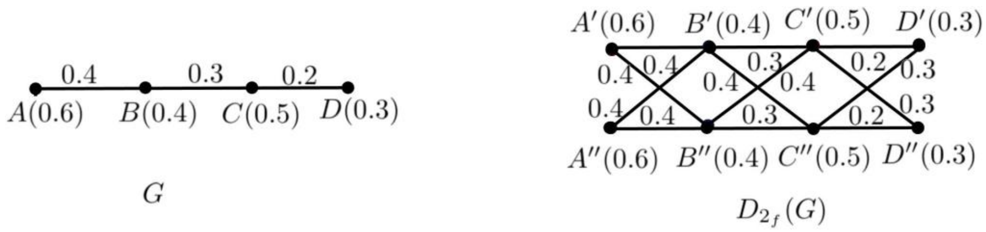

Figure 13. Fuzzy graph G and its shadow graph $D_{2_f}(G)$.

If a hub u fails due to a natural disaster, then the communication links between the failed hub u and its neighbouring hubs v_i become disrupted. As a result, the reliability of the communication links from u to its neighbouring hubs v_i is significantly reduced. Let $D_u \in [0,1]$ be the degradation factor, representing the percentage decline in the reliability of the hub u. If the degradation factor satisfies

$$D_u > \max \left\{ 1 - \frac{\frac{1}{2}(\sigma(u) \wedge \sigma(v_i))}{\mu(u, v_i)} \right\}, \quad (1)$$

then the reliability of each communication link from u to its neighbouring hubs v_i will be reduced, and each communication link will become weak with degraded membership values given by

$$\mu_D(u, v_i) = (1 - D_u).\mu(u, v_i). \quad (2)$$

If the degradation condition is not satisfied, then each communication link remains strong and functions normally despite the natural disaster.

Suppose the reliability of a hub B' is degraded by 55% $(i.e., D_B' = 0.55)$ due to a natural disaster. As a result, the communication link between A' and C' is disrupted, and its reliability will be significantly reduced. Since, D_B' satisfying (1), the edges $(B', A'), (B', A''), (B', C')$, and (B', C'') become weak, with membership values $0.18, 0.18, 0.13$, and 0.18, respectively (by (2)). Consequently, the communication links from the affected hub to its neighbouring hubs fail. Therefore, in this scenario, it is necessary to restore the failed connection to enable conflict-free communication between the hubs. To preserve the connectivity between A' and C', the corresponding backup hub B'' should be utilized to ensure reliable and interference-free communication. Subsequently, it is essential to allocate the minimum number of channels to maintain interference-free communication between the hubs. For that, we implement perfect fuzzy coloring on $D_{2_f}(G)$ to determine the minimum number of channels needed to maintain interference-free communication. The perfect fuzzy coloring of D_{2_f} is as follows:

Let G be the given fuzzy graph and D_{2_f} be its shadow graph. First, consider the vertex A' and color it with an arbitrary basic color, say $(R, 1)$. As the edge (A', B') is weak, the coloring of the vertex B' depends on the strength of the edge (A', B'). i.e., the membership value of vertex B' is $1 - I(A', B')$. Then the vertex B' will receive a fuzzy color $(R, 0.55)$. Similarly the vertex C' will receive a fuzzy color $(R, 0.67)$, the vertex A'' will receive a fuzzy color $(R, 0.55)$ and the vertex C'' will receive a fuzzy color $(R, 0.55)$. Now consider the vertex B' for coloring. Since all the incident edges are strong, it will receive a basic color $(B, 1)$. Similarly, the vertices B'' and D'' are also will receive the basic color $(B, 1)$. Thus, $D_{2_f}(G)$ is colored using two colors, namely Red and Blue.

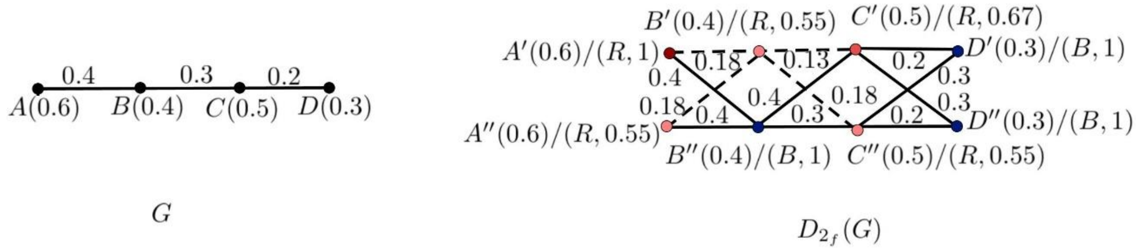

Figure 14. Fuzzy coloring of $D_{2_f}(G)$, when B' fails.

Therefore, $\chi_f(D_{2_f}(G)) = 2$. i.e., only two distinct channels are required to achieve conflict-free routing and ensure interference-free communication between the hubs.

Similarly, each fuzzy graph defined in this paper offers a versatile tool for analyzing and improving real-world networks.

5. Conclusion

In this paper, we precisely defined the concepts of the middle graph, splitting graph, shadow graph, line graph, total graph, and the subdivision graph of a fuzzy graph G. The chromatic numbers of the above-mentioned graphs, derived from the fuzzy paths, are determined by using fuzzy coloring based on the strength of the edges incident on each vertex.

Future work will focus on extending this analysis to the corresponding graphs of the fuzzy cycles and determining their chromatic numbers.

7. References

[1] Kannan, M., Sathiragavan, M., Nivetha, P., Sankar, K., and Gurjar, J., 2024, "Graph Coloring Techniques in Scheduling and Resource Allocation," *J. Nonlinear Anal. Optim.*, **15**(2).

[2] Karunambigai, M. G., and Mathew, J. A., 2024, "The Chromatic Number of Certain Families of Fuzzy Graphs," *Cuest. Fisioter.*, **53**(3), pp. 5153–5168.

[3] Munoz, S., Ortuno, M. T., Ramirez, J., and Yanez, J., 2005, "Coloring Fuzzy Graphs," *Omega*, **33**(3), pp. 211–221.

[4] Eslahchi, C., and Onagh, B. N., 2006, "Vertex-Strength of Fuzzy Graphs," *Int. J. Math. Math. Sci.*, **2006**(1), p. 043614.

[5] Samanta, S., Pramanik, T., and Pal, M., 2016, "Fuzzy Colouring of Fuzzy Graphs," *Afr. Mat.*, **27**, pp. 37–50.

[6] Mathew, S., Mordeson, J. N., and Malik, D. S., 2018, *Fuzzy Graph Theory*, Springer International Publishing, Berlin, Germany.

[7] Pal, M., Samanta, S., and Ghorai, G., 2020, "Fuzzy Cut Vertices and Fuzzy Trees," in *Modern Trends in Fuzzy Graph Theory*, Springer, Berlin, Ch. 3, pp. 115–124.

[8] Chithra, K. P., and Pilakkat, R., 2019, "Strength of Cartesian Product of Certain Strong Fuzzy Graphs," *Palest. J. Math.*, **8**(1).

[9] Gong, Z., and Zhang, J., 2022, "Chromatic Number of Fuzzy Graphs: Operations, Fuzzy Graph Coloring, and Applications," *Axioms*, **11**(12), p. 697.

[10] Karunambigai, M. G., and Muthusamy, A., 2006, "On Resolvable Multipartite G-Designs II," Graphs and Combinatorics, 22(1), pp. 59–67.