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On Vector Basis S-Cordial Graph

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Abstract
Let G be a (p, q) graph. Let V be an inner product space with basis S. We denote the inner product of the vectors x and y by $< x, y >$. Let $\varphi: V(G) \to S$ be a function. For each edge uv assign the label$< \varphi(u), \varphi(v) >$. We say that φ is a vector basis S-cordial labeling if $|\varphi_x - \varphi_y| \leq 1$ and $|\gamma_i - \gamma_j| \leq 1$ where $\varphi_x$denotes the number of vertices labeled with the vector x and γ_i denotes the number of edges labeled with the scalar i. A graph with a vector basis S-cordial labeling is called a vector basis S-cordial graph. In this paper, we investigate the vector basis$S$-cordial labeling behavior of corona product of alternate generalized snake graph with m copies of K_1 graph where $S = \{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$ is a basis in R^4.
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INTRODUCTION

In this paper, we consider only finite, simple and undirected graph $G = (V(G), E(G))$ where $V(G)$ and $E(G)$ respectively, denote the vertex set and edge set of G. Note that $p = |V(G)|$ and $q = |E(G)|$ denote the number of vertices and edges of G respectively. The idea of graph labeling was first introduced by Rosa in 1967 [11]. For a dynamic survey on graph labeling, we refer to Gallian [3]. Total edge irregularity strength for some snake related graphs was discussed in [1,15,16].The heronian mean labeling of graphs was introduced by Santhiya et al. [12] and also proved that the path, cycle, comb, dragon, triangular snake, quadrilateral snake, double triangular snake double quadrilateral snake, triple triangular snake, triple quadrilateral snake, star $K_{1,n}, n \leq 5$ and complete graph K_n, $n \leq 4$ admits heronian mean labeling [12, 13, 14]. The concept of cordial labeling was first introduced by Cahit [2]. For the terminologies and different notations of graph theory, we refer the book of Harary [4]and of algebra, we refer the book of Herstein [5]. Pair mean cordial labeling of triangular snake, alternate triangular snake, quadrilateral snake and alternate quadrilateral snake, hexagonal snake, irregular quadrilateral snake and triple triangular snake were discussed in [9,10]. We state a few definitions which are needed for proving the main results.

Definition 1.1. The generalized snake graph $GS_{n,k}$ is obtained from the path P_n by replaced each edge of the path by the cycle C_k.

Definition 1.2. The corona $G_1 \odot G_2$ of two graphs $G_1$and G_2 is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining each vertex of the i^{th} copy of G_2 to the i^{th} vertex of G_1.

Definition 1.3. The generalized alternate snake graph $GAS_{n,k}$ is obtained from the path P_n by replaced each alternate edge of the path by the cycle C_k.We have introduced new labeling technique called vector basis S-cordial labeling in [6] and investigated the vector basis vector basis $\{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$-cordial labelling behavior of some standard graphs like path, cycle, comb, star, complete graph, generalized friendship graph, tadpole graph, gear graph and thorn related graphs in [6-8]. In this paper, we investigate the vector basis $\{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$-cordial labeling behavior of corona product of generalized alternate snake graph with m copies of K_1 graph.

In this paper, we consider the inner product space R^n and the standard inner product $< x, y > = x_1y_1 + x_2y_2 + \cdots + x_ny_n$where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, $x_i, y_i \in R$.

Main Results

In this paper, we consider the generalized alternate snake graph with the pendent edges to have a cycle.

Theorem 2.1. The corona product of generalized alternate snake graph with m-copies of K_1, $AGS_{n,N} \odot mK_1$is a vector basis$\{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$-cordial for all $n \geq 2, N \geq 3$ and $m \geq 1$.

Proof. Let $V(AGS_{n,N} \odot mK_1) = \{u_{i,j}, u_{i,j}{}^k | 1 \le i \le \frac{n}{2},\ 1 \le j \le N \ and\ 1 \le k \le m\}$ and $E(AGS_{n,N} \odot mK_1) = \{u_{i,1}u_{i,N}, u_{i,j}u_{i,j+1} | 1 \le i \le \frac{n}{2},\ 1 \le j \le N-1\} \cup \{u_{i,N}u_{i+1,1} | 1 \le i \le \frac{n}{2}-1\} \cup \{u_{i,j}u_{i,j}{}^k | 1 \le i \le \frac{n}{2},\ 1 \le j \le N \ and\ 1 \le k \le m\}$ respectively be the vertex and edge sets of $AGS_{n,N} \odot mK_1$. Then $|V(AGS_{n,N} \odot mK_1)| = p = \frac{n}{2}N(m+1)$and $|E(AGS_{n,N} \odot mK_1)| = q = \frac{n}{2}(N(m+1)+1)-1$. Let us assign the vectors to the vertices in the following order $u_{11}, u_{12}, \dots, u_{1N}, \dots, u_{\frac{n}{2},1}, u_{\frac{n}{2},2}, \dots, u_{\frac{n}{2},N}, u_{11}{}^1, u_{11}{}^2, \dots, u_{11}{}^m, \dots, u_{1N}{}^1, u_{1N}{}^2, \dots, u_{1N}{}^m, \dots, u_{\frac{n}{2}1}{}^1, u_{\frac{n}{2}1}{}^2, \dots, u_{\frac{n}{2}1}{}^m, \dots, u_{\frac{n}{2}N}{}^1, u_{\frac{n}{2}N}{}^2, \dots, u_{\frac{n}{2}N}{}^m$. There are four case arises. Let $M = \frac{n}{2}$. Then $p = MN(m+1)$ and $q = MN(m+1) + M - 1$.

Case ($\boldsymbol{i}$): $M \equiv 0\ (mod\ 4)$

Then $M = 4t_1, t_1 > 0$.

Subcase ($\boldsymbol{I}$): $N \equiv 0\ (mod\ 4)$

Then, $N = 4t_2,\ t_2 > 0$. We assign the vector $(1,1,1,1)$ to the first $4t_1t_2$ vertices. Then assign the vector $(1,1,1,0)$ to the next $4t_1t_2$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2$vertices. Assign the vector $(1,0,0,0)$ to the next $4t_1t_2$ vertices.

Subcase ($\boldsymbol{I\ a}$): $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2)$ and $q = 4(16t_1t_2t_3 + 4t_1t_2 + t_1) - 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3$ pendent vertices. Then assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3$pendent vertices. Assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 4t_1t_2$. Thus $\gamma_4 = 16t_1t_2t_3 + 4t_1t_2 + t_1 - 1$ and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 4t_1t_2 + t_1$.

Subcase ($\boldsymbol{I\ b}$): $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \ge 0$. We obtain $p = 4(16t_1t_2t_3 + 8t_1t_2)$and $q = 4(16t_1t_2t_3 + 8t_1t_2 + t_1) - 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 8t_1t_2$. Thus $\gamma_4 = 16t_1t_2t_3 + 8t_1t_2 + t_1 - 1$ and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 8t_1t_2 + t_1$.

Subcase ($\boldsymbol{I\ c}$): $m \equiv 2\ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \ge 0$. We get $p = 4(16t_1t_2t_3 + 12t_1t_2)$and $q = 4(16t_1t_2t_3 + 12t_1t_2 + t_1) - 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_2$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 12t_1t_2$. Thus $\gamma_4 = 16t_1t_2t_3 + 8t_1t_2 + t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 12t_1t_2 + t_1$.

Subcase ($\boldsymbol{I\ d}$): $m \equiv 3\ (mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \ge 0$. We have $p = 4(16t_1t_2t_3 + 16t_1t_2)$ and $q = 4(16t_1t_2t_3 + 16t_1t_2 + t_1) - 1$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 16t_1t_2$. Thus $\gamma_4 = 16t_1t_2t_3 + 16t_1t_2 + t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 16t_1t_2 + t_1$.

Subcase ($\boldsymbol{II}$): $N \equiv 1\ (mod\ 4)$

Then, $N = 4t_2 + 1,\ t_2 > 0$. We assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + t_1$ vertices and assign the vector $(1,1,1,0)$ to the next$4t_1t_2$+$t_1$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2$+t_1 vertices and assign the vector $(1,0,0,0)$ to the next $4t_1t_2$+t_1 vertices.

Subcase ($\boldsymbol{II\ a}$): $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + t_1)$ and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 2t_1) - 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_3$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next$16t_1t_2t_3 + 4t_1t_3$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_3$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 2t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 2t_1$.

Subcase ($\boldsymbol{II\ b}$): $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 2t_1)$and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 3t_1) - 1$. Thereafter assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + t_1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + t_1$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + t_1$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + t_1$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 2t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 3t_1 - 1$ and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 3t_1$.

Subcase ($\boldsymbol{II\ c}$): $m \equiv 2\ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$. We obtain $p = 4(16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 3t_1)$ and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 4t_1) - 1$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 2t_1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 2t_1$pendent vertices. Also assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 2t_1$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 2t_1$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 3t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 4t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 4t_1$.

Subcase ($\boldsymbol{II\ d}$): $m \equiv 3(mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$. We obtain $p = 4(16t_1t_2t_3 + 16t_1t_2 + 4t_1t_3 + 4t_1)$ and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 4t_1t_3 + 5t_1) - 1$. In this case, assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 3t_1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 3t_1$pendent vertices. Also assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 3t_1$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 3t_1$pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 16t_1t_2 + 4t_1t_3 + 4t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 16t_1t_2 + 4t_1t_3 + 5t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 16t_1t_2 + 4t_1t_3 + 5t_1$.

Subcase ($\boldsymbol{III}$): $N \equiv 2\ (mod\ 4)$

Then, $N = 4t_2 + 2,\ t_2 > 0$. We assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + 2t_1$ vertices and assign the vector $(1,1,1,0)$ to the next $4t_1t_2+2t_1$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2+2t_1$vertices and assign the vector $(1,0,0,0)$ to the next $4t_1t_2+2t_1$ vertices.

Subcase ($\boldsymbol{III\ a}$): $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 2t_1)$ and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 3t_1) - 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_3$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_3$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_3$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 2t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 3t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 3t_1$.

Subcase ($\boldsymbol{III\ b}$): $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 4t_1)$ and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 5t_1) - 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 2t_1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 2t_1$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 2t_1$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 2t_1$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 4t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 5t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 5t_1$.

Subcase ($\boldsymbol{III\ c}$): $m \equiv 2\ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$. We have $p = 4(16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 6t_1)$ and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 7t_1) - 1$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 4t_1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 4t_1$pendent vertices. Also assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 4t_1$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 4t_1$pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} =$

$\varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 6t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 7t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 7t_1$.

Subcase $(\boldsymbol{III\ d})$: $m \equiv 3\ (mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 16t_1t_2 + 8t_1t_3 + 8t_1)$ and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 8t_1t_3 + 9t_1) - 1$. In this case assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 6t_1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 6t_1$pendent vertices. Also assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 6t_1$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 6t_1$pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 16t_1t_2 + 8t_1t_3 + 8t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 16t_1t_2 + 8t_1t_3 + 9t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 16t_1t_2 + 8t_1t_3 + 9t_1$.

Subcase $(\boldsymbol{IV})$: $N \equiv 3\ (mod\ 4)$

Then, $N = 4t_2 + 3,\ t_2 \geq 0$. We assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + 3t_1$ vertices and assign the vector $(1,1,1,0)$ to the next $4t_1t_2$+$3t_1$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2$+$3t_1$vertices and assign the vector $(1,0,0,0)$ to the next $4t_1t_2$+$3t_1$ vertices.

Subcase $(\boldsymbol{IV\ a})$: $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 3t_1)$ and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 4t_1) - 1$. We now assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_3$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_3$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_3$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 3t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 4t_1 - 1$ and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 4t_1$.

Subcase $(\boldsymbol{IV\ b})$: $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 6t_1)$ and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 7t_1) - 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 3t_1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 3t_1$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 3t_1$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 3t_1$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 6t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 7t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 7t_1$.

Subcase $(\boldsymbol{IV\ c})$: $m \equiv 2\ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$. We have $p = 4(16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 9t_1)$ and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 10t_1) - 1$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 6t_1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 6t_1$pendent vertices. Also assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 6t_1$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 6t_1$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 9t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 10t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 10t_1$.

Subcase $(\boldsymbol{IV\ d})$: $m \equiv 3\ (mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 12t_1)$ and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 13t_1) - 1$. In this case assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 9t_1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 9t_1$pendent vertices. Also assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 9t_1$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 9t_1$pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 12t_1$. Thus $\gamma_4 = 16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 13t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 13t_1$.

Case $(\boldsymbol{ii})$: $M \equiv 1\ (mod\ 4)$

Then $M = 4t_1 + 1, t_1 \geq 0$.

Subcase $(\boldsymbol{I})$: $N \equiv 0\ (mod\ 4)$

Then, $N = 4t_2,\ t_2 > 0$. We assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + 4t_2$ vertices. Then assign the vector $(1,1,1,0)$ to the next $4t_1t_2$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2$vertices. Assign the vector $(1,0,0,0)$ to the next $4t_1t_2$ vertices.

Subcase $(\boldsymbol{I\ a})$: $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2 + 4t_2t_3 + t_2)$ and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 4t_2t_3 + t_2 + t_1)$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_2t_3 - 3t_2$ pendent vertices. Then assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_2t_3 + t_2$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_2t_3 + t_2$pendent vertices. Assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_2t_3 + t_2$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 4t_1t_2 + 4t_2t_3 + t_2$.Thus $\gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 4t_1t_2 + 4t_2t_3 + t_2 + t_1$.

Subcase $(\boldsymbol{I\ b})$: $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$. We obtain $p = 4(16t_1t_2t_3 + 8t_1t_2 + 4t_2t_3 + 2t_2)$ and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 4t_2t_3 + 2t_2 + t_1)$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 4t_2t_3 - 2t_2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_2t_3 + 2t_2$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_2t_3 + 2t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_2t_3 + 2t_2$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 8t_1t_2 + 4t_2t_3 + 2t_2$. Thus $\gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 8t_1t_2 + 4t_2t_3 + 2t_2 + t_1$.

Subcase $(\boldsymbol{I\ c})$: $m \equiv 2\ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 12t_1t_2 + 4t_2t_3 + 3t_2)$ and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 4t_2t_3 + 3t_2 + t_1)$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_2 + 4t_2t_3 - t_2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_2t_3 + 3t_2$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_2t_3 + 3t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_2t_3 + 3t_2$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 12t_1t_2 + 4t_2t_3 + 3t_2$ .Thus $\gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 12t_1t_2 + 4t_2t_3 + 3t_2 + t_1$.

Subcase $(\boldsymbol{I\ d})$: $m \equiv 3\ (mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$. We have $p = 4(16t_1t_2t_3 + 16t_1t_2 + 4t_2t_3 + 4t_2)$ and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 4t_2t_3 + 4t_2 + t_1)$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 16t_1t_2 + 4t_2t_3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 16t_1t_2 + 4t_2t_3 + 4t_2$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 16t_1t_2 + 4t_2t_3 + 4t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 16t_1t_2 + 4t_2t_3 + 4t_2$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 16t_1t_2 + 4t_2t_3 + 4t_2$. Thus $\gamma_4 = 16t_1t_2t_3 + 16t_1t_2 + t_1 - 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 16t_1t_2 + 4t_2t_3 + 4t_2 + t_1$.

Subcase $(\boldsymbol{II})$: $N \equiv 1\ (mod\ 4)$

Then, $N = 4t_2 + 1,\ t_2 > 0$. We assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + t_1 + 4t_2 + 1$ vertices and assign the vector $(1,1,1,0)$ to the next $4t_1t_2+t_1$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2+t_1$vertices and assign the vector $(1,0,0,0)$ to the next $4t_1t_2+t_1$ vertices.

Subcase $(\boldsymbol{II\ a})$: $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 4t_2t_3 + t_1 + t_2 + t_3) + 1$and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 4t_2t_3 + 2t_1 + t_2 + t_3) + 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_3 + 4t_2t_3 - 3t_2 + t_3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_3 + 4t_2t_3 + t_2 + t_3$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_3 + 4t_2t_3 + t_2 + t_3$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_3 + 4t_2t_3 + t_2 + t_3$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = 16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 4t_2t_3 + t_1 + t_2 + t_3 + 1$ and $\varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 4t_2t_3 + t_1 + t_2 + t_3$.Thus $\gamma_4 = 16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 4t_2t_3 + 2t_1 + t_2 + t_3 + 1$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 4t_2t_3 + 2t_1 + t_2 + t_3$.

Subcase $(\boldsymbol{II\ b})$: $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 4t_2t_3 + 2t_1 + 2t_2 + t_3) + 2$ and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 4t_2t_3 + 3t_1 + 2t_2 + t_3) + 1$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 4t_2t_3 + t_1 - 2t_2 + t_3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 4t_2t_3 + t_1 + 2t_2 + t_3$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 4t_2t_3 + t_1 + 2t_2 + t_3$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 4t_2t_3 + t_1 + 2t_2 + t_3$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = 16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 4t_2t_3 + 2t_1 + 2t_2 + t_3 + 1 = \varphi_{(1,0,0,0)}$ and $\varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = 16t_1t_2t_3 + 8t_1t_2 +$

$4t_1t_3+4t_2t_3+2t_1+2t_2+t_3$. Thus $\gamma_4=\gamma_1=16t_1t_2t_3+8t_1t_2+4t_1t_3+4t_2t_3+3t_1+2t_2+t_3+1$and $\gamma_3=\gamma_2=16t_1t_2t_3+8t_1t_2+4t_1t_3+4t_2t_3+3t_1+2t_2+t_3$.
Subcase ($\boldsymbol{II\ c}$): $m\equiv 2\ (mod\ 4)$
Then, $m=4t_3+2,\ t_3\geq 0$. We obtain $p=4(16t_1t_2t_3+12t_1t_2+4t_1t_3+4t_2t_3+3t_1+3t_2+t_3)+3$and $q=4(16t_1t_2t_3+12t_1t_2+4t_1t_3+4t_2t_3+4t_1+3t_2+t_3)+3$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3+8t_1t_2+4t_1t_3+4t_2t_3+2t_1-t_2+t_3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3+8t_1t_2+4t_1t_3+4t_2t_3+2t_1+3t_2+t_3$pendent vertices. Also assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3+8t_1t_2+4t_1t_3+4t_2t_3+2t_1+3t_2+t_3+1$ pendent vertices and assign the vector$(1,0,0,0)$ to the next $16t_1t_2t_3+8t_1t_2+4t_1t_3+4t_2t_3+2t_1+3t_2+t_3+1$ pendent vertices. Hence $\varphi_{(1,1,1,1)}=\varphi_{(1,1,0,0)}=\varphi_{(1,0,0,0)}=16t_1t_2t_3+12t_1t_2+4t_1t_3+4t_2t_3+3t_1+3t_2+t_3+1$ and $\varphi_{(1,1,1,0)}=16t_1t_2t_3+12t_1t_2+4t_1t_3+4t_2t_3+3t_1+3t_2+t_3$ Thus $\gamma_4=\gamma_2=\gamma_1=16t_1t_2t_3+12t_1t_2+4t_1t_3+4t_2t_3+4t_1+3t_2+t_3+1$and $\gamma_3=16t_1t_2t_3+12t_1t_2+4t_1t_3+4t_2t_3+4t_1+3t_2+t_3$.
Subcase ($\boldsymbol{II\ d}$): $m\equiv 3(mod\ 4)$
Then, $m=4t_3+3,\ t_3\geq 0$. We have $p=4(16t_1t_2t_3+16t_1t_2+4t_1t_3+4t_2t_3+4t_1+4t_2+t_3+1)$and $q=4(16t_1t_2t_3+16t_1t_2+4t_1t_3+4t_2t_3+5t_1+4t_2+t_3+1)$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3+12t_1t_2+4t_1t_3+4t_2t_3+4t_1+t_3+1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3+12t_1t_2+4t_1t_3+4t_2t_3+4t_1+4t_2+t_3+1$ pendent vertices. Also assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3+12t_1t_2+4t_1t_3+4t_2t_3+4t_1+4t_2+t_3+1$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3+12t_1t_2+4t_1t_3+4t_2t_3+4t_1+4t_2+t_3+1$ pendent vertices. Hence $\varphi_{(1,1,1,1)}=\varphi_{(1,1,1,0)}=\varphi_{(1,1,0,0)}=\varphi_{(1,0,0,0)}=16t_1t_2t_3+16t_1t_2+4t_1t_3+4t_2t_3+4t_1+4t_2+t_3+1$.
Thus $\gamma_4=\gamma_3=\gamma_2=\gamma_1=16t_1t_2t_3+16t_1t_2+4t_1t_3+4t_2t_3+5t_1+4t_2+t_3+1$.
Subcase ($\boldsymbol{III}$): $N\equiv 2\ (mod\ 4)$
Then, $N=4t_2+2,\ t_2>0$. We assign the vector $(1,1,1,1)$ to the first $4t_1t_2+2t_1+4t_2+2$ vertices and assign the vector $(1,1,1,0)$ to the next $4t_1t_2+2t_1$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2+2t_1$vertices and assign the vector $(1,0,0,0)$ to the next $4t_1t_2+2t_1$ vertices.
Subcase ($\boldsymbol{III\ a}$): $m\equiv 0\ (mod\ 4)$
Then, $m=4t_3,\ t_3>0$. We have $p=4(16t_1t_2t_3+4t_1t_2+8t_1t_3+4t_2t_3+2t_1+t_2+2t_3)+2$and $=4(16t_1t_2t_3+4t_1t_2+8t_1t_3+4t_2t_3+3t_1+t_2+2t_3)+2$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3+8t_1t_3+4t_2t_3-3t_2+2t_3-1$pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3+8t_1t_3+4t_2t_3+t_2+2t_3$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3+8t_1t_3+4t_2t_3+t_2+2t_3$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3+8t_1t_3+4t_2t_3+t_2+2t_3+1$ pendent vertices. Hence $\varphi_{(1,1,1,1)}=\varphi_{(1,0,0,0)}=16t_1t_2t_3+4t_1t_2+8t_1t_3+4t_2t_3+2t_1+t_2+2t_3+1$and $\varphi_{(1,1,1,0)}=\varphi_{(1,1,0,0)}=16t_1t_2t_3+4t_1t_2+8t_1t_3+4t_2t_3+2t_1+t_2+2t_3$. Thus $\gamma_4=\gamma_1=16t_1t_2t_3+4t_1t_2+8t_1t_3+4t_2t_3+3t_1+t_2+2t_3+1$and $\gamma_3=\gamma_2=16t_1t_2t_3+4t_1t_2+8t_1t_3+4t_2t_3+3t_1+t_2+2t_3$.
Subcase ($\boldsymbol{III\ b}$): $m\equiv 1(mod\ 4)$
Then, $m=4t_3+1,\ t_3\geq 0$. We obtain $p=4(16t_1t_2t_3+8t_1t_2+8t_1t_3+4t_2t_3+4t_1+2t_2+2t_3+1)$and $=4(16t_1t_2t_3+8t_1t_2+8t_1t_3+4t_2t_3+5t_1+t_2+2t_3+1)$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3+4t_1t_2+8t_1t_3+4t_2t_3+2t_1-2t_2+2t_3-1$pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3+4t_1t_2+8t_1t_3+4t_2t_3+2t_1+2t_2+2t_3+1$ pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3+4t_1t_2+8t_1t_3+4t_2t_3+2t_1+2t_2+2t_3+1$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3+4t_1t_2+8t_1t_3+4t_2t_3+2t_1+2t_2+2t_3+1$ pendent vertices. Hence $\varphi_{(1,1,1,1)}=\varphi_{(1,1,1,0)}=\varphi_{(1,1,0,0)}=\varphi_{(1,0,0,0)}=16t_1t_2t_3+8t_1t_2+8t_1t_3+4t_2t_3+4t_1+2t_2+2t_3+1$. Thus $\gamma_4=\gamma_3=\gamma_2=\gamma_1=16t_1t_2t_3+8t_1t_2+8t_1t_3+4t_2t_3+5t_1+2t_2+2t_3+1$.
Subcase ($\boldsymbol{III\ c}$): $m\equiv 2(mod\ 4)$
Then, $m=4t_3+2,\ t_3\geq 0$. We obtain $p=4(16t_1t_2t_3+12t_1t_2+8t_1t_3+4t_2t_3+6t_1+3t_2+2t_3+1)+2$and $q=4(16t_1t_2t_3+12t_1t_2+8t_1t_3+4t_2t_3+7t_1+3t_2+2t_3+1)+2$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3+8t_1t_2+8t_1t_3+4t_2t_3+4t_1-t_2+2t_3$pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3+8t_1t_2+8t_1t_3+4t_2t_3+4t_1+3t_2+2t_3+1$ pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3+8t_1t_2+8t_1t_3+4t_2t_3+4t_1+3t_2+2t_3+1$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3+8t_1t_2+8t_1t_3+4t_2t_3+4t_1+3t_2+2t_3+2$ pendent vertices. Hence $\varphi_{(1,1,1,1)}=\varphi_{(1,0,0,0)}=16t_1t_2t_3+12t_1t_2+8t_1t_3+4t_2t_3+6t_1+3t_2+2t_3+2$and $\varphi_{(1,1,1,0)}=\varphi_{(1,1,0,0)}=16t_1t_2t_3+12t_1t_2+8t_1t_3+4t_2t_3+6t_1+3t_2+2t_3+1$. Thus $\gamma_4=\gamma_1=$

$16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 4t_2t_3 + 7t_1 + 3t_2 + 2t_3 + 2$and $\gamma_3 = \gamma_2 = 16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 4t_2t_3 + 7t_1 + 3t_2 + 2t_3 + 1$.

Subcase ($III\ d$): $m \equiv 3(mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$. We obtain $p = 4(16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 4t_2t_3 + 8t_1 + 4t_2 + 2t_3 + 2)$and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 8t_1t_3 + 4t_2t_3 + 8t_1 + 4t_2 + 2t_3 + 2)$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 4t_2t_3 + 6t_1 + 2t_3$pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 4t_2t_3 + 6t_1 + 4t_2 + 2t_3 + 2$ pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 4t_2t_3 + 6t_1 + 4t_2 + 2t_3 + 2$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 4t_2t_3 + 6t_1 + 4t_2 + 2t_3 + 2$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 16t_1t_2 + 8t_1t_3 + 4t_2t_3 + 8t_1 + 4t_2 + 2t_3 + 2$. Thus $\gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 16t_1t_2 + 8t_1t_3 + 4t_2t_3 + 9t_1 + 4t_2 + 2t_3 + 2$.

Subcase (IV): $N \equiv 3\ (mod\ 4)$

Then, $N = 4t_2 + 3,\ t_2 > 0$. We assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + 3t_1 + 4t_2 + 3$ vertices and assign the vector $(1,1,1,0)$ to the next $4t_1t_2 + 3t_1$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2 + 3t_1$vertices and assign the vector $(1,0,0,0)$ to the next $4t_1t_2 + 3t_1$ vertices.

Subcase ($IV\ a$): $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 4t_2t_3 + 3t_1 + t_2 + 3t_3) + 3$and $= 4(16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 4t_2t_3 + 4t_1 + t_2 + 3t_3) + 3$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_3 + 4t_2t_3 - 3t_2 + 3t_3 - 2$pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 4t_2t_3 + t_2 + 3t_3$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 4t_2t_3 + t_2 + 3t_3 + 1$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 4t_2t_3 + t_2 + 3t_3 + 1$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,0,0,0)} = \varphi_{(1,1,0,0)} = 16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 4t_2t_3 + 3t_1 + t_2 + 3t_3 + 1$and $\varphi_{(1,1,1,0)} = 16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 4t_2t_3 + 3t_1 + t_2 + 3t_3$. Thus $\gamma_4 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 4t_2t_3 + 4t_1 + t_2 + 3t_3 + 1$and $\gamma_3 = 16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 4t_2t_3 + 3t_1 + t_2 + 2t_3$.

Subcase ($IV\ b$): $m \equiv 1(mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$. We obtain $p = 4(16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 4t_2t_3 + 6t_1 + 2t_2 + 3t_3 + 1) + 2$and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 4t_2t_3 + 7t_1 + 2t_2 + 3t_3 + 1) + 2$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 4t_2t_3 + 3t_1 - 2t_2 + 3t_3 - 1$pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 4t_2t_3 + 3t_1 + 2t_2 + 3t_3 + 1$ pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 4t_2t_3 + 3t_1 + 2t_2 + 3t_3 + 1$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 4t_2t_3 + 3t_1 + 2t_2 + 3t_3 + 2$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 4t_2t_3 + 6t_1 + 2t_2 + 3t_3 + 2$and $\varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = 16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 4t_2t_3 + 6t_1 + 2t_2 + 3t_3 + 1$. Thus $\gamma_4 = \gamma_1 = 16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 4t_2t_3 + 7t_1 + 2t_2 + 3t_3 + 2$and $\gamma_3 = \gamma_2 = 16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 4t_2t_3 + 7t_1 + 2t_2 + 3t_3 + 1$.

Subcase ($IV\ c$): $m \equiv 2(mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$. We obtain $p = 4(16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 4t_2t_3 + 9t_1 + 3t_2 + 3t_3 + 2) + 1$and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 4t_2t_3 + 10t_1 + 3t_2 + 3t_3 + 2) + 1$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 4t_2t_3 + 6t_1 - t_2 + 3t_3$pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 4t_2t_3 + 6t_1 + 3t_2 + 3t_3 + 2$ pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 4t_2t_3 + 6t_1 + 3t_2 + 3t_3 + 2$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 4t_2t_3 + 6t_1 + 3t_2 + 3t_3 + 2$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = 16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 4t_2t_3 + 9t_1 + 3t_2 + 3t_3 + 3$and $\varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 4t_2t_3 + 9t_1 + 3t_2 + 3t_3 + 2$. Thus $\gamma_4 = 16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 4t_2t_3 + 10t_1 + 3t_2 + 3t_3 + 3$and $\gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 4t_2t_3 + 10t_1 + 3t_2 + 3t_3 + 2$.

Subcase ($IV\ d$): $m \equiv 3(mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$. We obtain $p = 4(16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 4t_2t_3 + 12t_1 + 4t_2 + 3t_3 + 3)$and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 4t_2t_3 + 13t_1 + 4t_2 + 3t_3 + 3)$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 4t_2t_3 + 9t_1 + 3t_3$pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 4t_2t_3 + 9t_1 + 4t_2 + 3t_3 + 3$ pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 4t_2t_3 + 9t_1 + 4t_2 + 3t_3 +$

3 pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 4t_2t_3 + 9t_1 + 4t_2 + 3t_3 + 3$ pendent vertices. Hence $\varphi_{(1,1,1,1)} = \varphi_{(1,1,1,0)} = \varphi_{(1,1,0,0)} = \varphi_{(1,0,0,0)} = 16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 4t_2t_3 + 12t_1 + 4t_2 + 3t_3 + 3$. Thus $\gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = 16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 4t_2t_3 + 13t_1 + 4t_2 + 3t_3 + 3$.

Case (iii): $M \equiv 2 \ (mod\ 4)$

Then $M = 4t_1 + 2, t_1 \geq 0$.

Subcase (I): $N \equiv 0 \ (mod\ 4)$

Then, $N = 4t_2,\ t_2 > 0$. We assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + 4t_2$ vertices and assign the vector $(1,1,1,0)$ to the next $4t_1t_2 + 4t_2$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2$vertices and assign the vector $(1,0,0,0)$ to the next $4t_1t_2$ vertices.

Subcase $(I\ a)$: $m \equiv 0 \ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2 + 8t_2t_3 + 2t_2)$and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 8t_2t_3 + 2t_2 + t_1) + 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_2t_3 - 2t_2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_2t_3 - 2t_2$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_2t_3 + 2t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_2t_3 + 2t_2$ pendent vertices.

Subcase $(I\ b)$: $m \equiv 1 \ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$. We obtain $p = 4(16t_1t_2t_3 + 8t_1t_2 + 8t_2t_3 + 4t_2)$and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 8t_2t_3 + 4t_2 + t_1) + 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 8t_2t_3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_2t_3$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_2t_3 + 4t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_2t_3 + 4t_2$ pendent vertices.

Subcase $(I\ c)$: $m \equiv 2 \ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$. We obtain $p = 4(16t_1t_2t_3 + 12t_1t_2 + 8t_2t_3 + 6t_2)$and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 8t_2t_3 + 6t_2 + t_1) + 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_2 + 8t_2t_3 + 2t_2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_2t_3 + 2t_2$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_2t_3 + 6t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_2t_3 + 6t_2$ pendent vertices.

Subcase $(I\ d)$: $m \equiv 3 \ (mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 16t_1t_2 + 8t_2t_3 + 8t_2)$and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 8t_2t_3 + 8t_2 + t_1) + 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2 + 8t_2t_3 + 4t_2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_2t_3 + 4t_2$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_2t_3 + 8t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_2t_3 + 8t_2$ pendent vertices.

Subcase (II): $N \equiv 1 \ (mod\ 4)$

Then, $N = 4t_2 + 1,\ t_2 > 0$. We assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + 4t_2 + t_1 + 1$ vertices and assign the vector $(1,1,1,0)$ to the next $4t_1t_2 + 4t_2 + t_1 + 1$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2 + t_1$vertices and assign the vector $(1,0,0,0)$ to the next $4t_1t_2 + t_1$ vertices.

Subcase $(II\ a)$: $m \equiv 0 \ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 8t_2t_3 + t_1 + 2t_2 + 2t_3) + 2$and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 8t_2t_3 + 2t_1 + 2t_2 + 2t_3) + 3$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_3 + 8t_2t_3 - 2t_2 + 2t_3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_3 + 8t_2t_3 - 2t_2 + 2t_3 - 1$ pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_3 + 8t_2t_3 + 2t_2 + 2t_3$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_3 + 8t_2t_3 + 2t_2 + 2t_3 + 1$pendent vertices.

Subcase $(II\ b)$: $m \equiv 1 \ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 8t_2t_3 + 2t_1 + 4t_2 + 2t_3 + 1)$and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 8t_2t_3 + 3t_1 + 4t_2 + 2t_3 + 1) + 1$. Assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 8t_2t_3 + t_1 + 2t_3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 8t_2t_3 + t_1 + 2t_3$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 8t_2t_3 + t_1 + 4t_2 + 2t_3 + 1$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 8t_2t_3 + t_1 + 4t_2 + 2t_3 + 1$ pendent vertices.

Subcase $(II\ c)$: $m \equiv 2 \ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$. We have $p = 4(16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 8t_2t_3 + 3t_1 + 6t_2 + 2t_3 + 1) + 2$and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 8t_2t_3 + 4t_1 + 6t_2 + 2t_3 + 1) + 3$. We assign the vector (1,1,1,1) to the first $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 8t_2t_3 + 2t_1 + 2t_2 + 2t_3 + 1$ pendent vertices and assign the vector (1,1,1,0) to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 8t_2t_3 + 2t_1 + 2t_2 + 2t_3$pendent vertices. Then assign the vector (1,1,0,0) to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 8t_2t_3 + 2t_1 + 6t_2 + 2t_3 + 1$ pendent vertices and assign the vector(1,0,0,0) to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 8t_2t_3 + 2t_1 + 6t_2 + 2t_3 + 2$pendent vertices.

Subcase ($\boldsymbol{II\ d}$): $m \equiv 3(mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$. We have $p = 4(16t_1t_2t_3 + 16t_1t_2 + 4t_1t_3 + 8t_2t_3 + 4t_1 + 8t_2 + 2t_3 + 2)$and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 4t_1t_3 + 8t_2t_3 + 5t_1 + 8t_2 + 2t_3 + 2) + 1$. Assign the vector (1,1,1,1) to the first $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 8t_2t_3 + 3t_1 + 4t_2 + 2t_3 + 1$ pendent vertices and assign the vector (1,1,1,0) to the next $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 8t_2t_3 + 3t_1 + 4t_2 + 2t_3 + 1$pendent vertices. Then assign the vector (1,1,0,0) to the next $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 8t_2t_3 + 3t_1 + 8t_2 + 2t_3 + 2$ pendent vertices and assign the vector (1,0,0,0) to the next $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 8t_2t_3 + 3t_1 + 8t_2 + 2t_3 + 2$pendent vertices.

Subcase ($\boldsymbol{III}$): $N \equiv 2\ (mod\ 4)$

Then, $N = 4t_2 + 2,\ t_2 > 0$. We assign the vector (1,1,1,1) to the first $4t_1t_2 + 4t_2 + 2t_1 + 2$ vertices and assign the vector (1,1,1,0) to the next $4t_1t_2 + 4t_2 + 2t_1 + 2$vertices. Also assign the vector (1,1,0,0) to the next $4t_1t_2 + 2t_1$vertices and assign the vector (1,0,0,0) to the next $4t_1t_2 + 2t_1$ vertices.

Subcase ($\boldsymbol{III\ a}$): $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 8t_2t_3 + 2t_1 + 2t_2 + 4t_3 + 1)$and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 8t_2t_3 + 3t_1 + 2t_2 + 4t_3 + 1) + 1$. We assign the vector (1,1,1,1) to the first $16t_1t_2t_3 + 8t_1t_3 + 8t_2t_3 - 2t_2 + 4t_3 - 1$ pendent vertices and assign the vector (1,1,1,0) to the next $16t_1t_2t_3 + 8t_1t_3 + 8t_2t_3 - 2t_2 + 4t_3 - 1$ pendent vertices. Then assign the vector (1,1,0,0) to the next $16t_1t_2t_3 + 8t_1t_3 + 8t_2t_3 + 2t_2 + 4t_3 + 1$ pendent vertices and assign the vector (1,0,0,0) to the next $16t_1t_2t_3 + 8t_1t_3 + 8t_2t_3 + 2t_2 + 4t_3 + 1$pendent vertices.

Subcase ($\boldsymbol{III\ b}$): $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 8t_2t_3 + 4t_1 + 4t_2 + 4t_3 + 2)$and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 8t_2t_3 + 5t_1 + 4t_2 + 4t_3 + 2) + 1$. We assign the vector (1,1,1,1) to the first $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 8t_2t_3 + 2t_1 + 4t_3$ pendent vertices and assign the vector (1,1,1,0) to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 8t_2t_3 + 2t_1 + 4t_3$pendent vertices. Then assign the vector (1,1,0,0) to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 8t_2t_3 + 2t_1 + 4t_2 + 4t_3 + 2$ pendent vertices and assign the vector (1,0,0,0) to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 8t_2t_3 + 2t_1 + 4t_2 + 4t_3 + 2$pendent vertices.

Subcase ($\boldsymbol{III\ c}$): $m \equiv 2\ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$. We have $p = 4(16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 8t_2t_3 + 6t_1 + 6t_2 + 4t_3 + 3)$and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 8t_2t_3 + 7t_1 + 6t_2 + 4t_3 + 3) + 1$. Then assign the vector (1,1,1,1) to the first $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 8t_2t_3 + 4t_1 + 2t_2 + 4t_3 + 1$ pendent vertices and assign the vector (1,1,1,0) to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 8t_2t_3 + 4t_1 + 2t_2 + 4t_3 + 1$pendent vertices. Now assign the vector (1,1,0,0) to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 8t_2t_3 + 4t_1 + 6t_2 + 4t_3 + 3$ pendent vertices and assign the vector (1,0,0,0) to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 8t_2t_3 + 4t_1 + 6t_2 + 4t_3 + 3$ pendent vertices.

Subcase ($\boldsymbol{III\ d}$): $m \equiv 3\ (mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 16t_1t_2 + 8t_1t_3 + 8t_2t_3 + 8t_1 + 8t_2 + 4t_3 + 4)$and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 8t_2t_3 + 9t_1 + 8t_2 + 4t_3 + 4) + 1$. Then assign the vector (1,1,1,1) to the first $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 8t_2t_3 + 6t_1 + 4t_2 + 4t_3 + 2$ pendent vertices and assign the vector (1,1,1,0) to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 8t_2t_3 + 6t_1 + 4t_2 + 4t_3 + 2$pendent vertices. Now assign the vector (1,1,0,0) to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 8t_2t_3 + 6t_1 + 8t_2 + 4t_3 + 2$ pendent vertices and assign the vector (1,0,0,0) to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 8t_2t_3 + 6t_1 + 8t_2 + 4t_3 + 2$ pendent vertices.

Subcase ($\boldsymbol{IV}$): $N \equiv 3\ (mod\ 4)$

Then, $N = 4t_2 + 3,\ t_2 \geq 0$. We assign the vector (1,1,1,1) to the first $4t_1t_2 + 4t_2 + 3t_1 + 3$ vertices and assign the vector (1,1,1,0) to the next $4t_1t_2 + 4t_2 + 3t_1 + 3$vertices. Also assign the vector (1,1,0,0) to the next $4t_1t_2 + 3t_1$vertices and assign the vector (1,0,0,0) to the next $4t_1t_2 + 3t_1$ vertices.

Subcase ($\boldsymbol{IV\ a}$): $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 8t_2t_3 + 3t_1 + 2t_2 + 6t_3) + 2$and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 8t_2t_3 + 4t_1 + 2t_2 + 6t_3) + 3$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_3 + 8t_2t_3 - 2t_2 + 6t_3 - 2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 8t_2t_3 - 2t_2 + 6t_3 - 3$ pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 8t_2t_3 + 2t_2 + 6t_3 + 1$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 8t_2t_3 + 2t_2 + 6t_3 + 1$pendent vertices.

Subcase ($\boldsymbol{IV\ b}$): $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 8t_2t_3 + 6t_1 + 4t_2 + 6t_3 + 1)$and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 8t_2t_3 + 7t_1 + 4t_2 + 6t_3 + 1) + 1$. Assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 8t_2t_3 + 3t_1 + 6t_3 - 2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 8t_2t_3 + 3t_1 + 6t_3 - 2$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 8t_2t_3 + 3t_1 + 4t_2 + 6t_3 + 1$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 8t_2t_3 + 3t_1 + 4t_2 + 6t_3 + 1$pendent vertices.

Subcase ($\boldsymbol{IV\ c}$): $m \equiv 2\ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 8t_2t_3 + 9t_1 + 6t_2 + 6t_3 + 4) + 2$and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 8t_2t_3 + 10t_1 + 6t_2 + 6t_3 + 4) + 3$. Assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_3 + 8t_2t_3 + 6t_1 - 2t_2 + 6t_3 + 1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 8t_2t_3 + 6t_1 - 2t_2 + 6t_3 + 1$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 8t_2t_3 + 6t_1 + 6t_2 + 6t_3 + 5$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 8t_2t_3 + 6t_1 + 6t_2 + 6t_3 + 5$pendent vertices.

Subcase ($\boldsymbol{IV\ d}$): $m \equiv 3\ (mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$. We have $p = 4(16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 8t_2t_3 + 12t_1 + 8t_2 + 6t_3 + 6)$and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 8t_2t_3 + 13t_1 + 8t_2 + 6t_3 + 6) + 1$. Then assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 8t_2t_3 + 9t_1 + 4t_2 + 6t_3 + 3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 8t_2t_3 + 9t_1 + 4t_2 + 6t_3 + 3$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 8t_2t_3 + 9t_1 + 8t_2 + 6t_3 + 6$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 8t_2t_3 + 9t_1 + 8t_2 + 6t_3 + 6$pendent vertices.

Case ($\boldsymbol{iv}$): $M \equiv 3\ (mod\ 4)$

Then $M = 4t_1 + 3, t_1 \geq 0$.

Subcase ($\boldsymbol{I}$): $N \equiv 0\ (mod\ 4)$

Then, $N = 4t_2,\ t_2 > 0$. We assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + 4t_2$ vertices. Then assign the vector $(1,1,1,0)$ to the next $4t_1t_2\ + 4t_2$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2 + 4t_2$vertices. Assign the vector $(1,0,0,0)$ to the next $4t_1t_2$ vertices.

Subcase ($\boldsymbol{I\ a}$): $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$. We have $p = 4(16t_1t_2t_3 + 4t_1t_2 + 12t_2t_3 + 3t_2)$and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 12t_2t_3 + 3t_2 + t_1) + 2$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_2t_3 - t_2$ pendent vertices. Then assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_2t_3 - t_2$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_2t_3 - t_2$pendent vertices. Assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_2t_3 + 3t_2$ pendent vertices.

Subcase ($\boldsymbol{I\ b}$): $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 8t_1t_2 + 12t_2t_3 + 6t_2)$and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 12t_2t_3 + 6t_2 + t_1) + 2$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 12t_2t_3 + 2t_2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_2t_3 + 2t_2$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_2t_3 + 2t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_2t_3 + 6t_2$ pendent vertices.

Subcase ($\boldsymbol{I\ c}$): $m \equiv 2\ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$. We have $p = 4(16t_1t_2t_3 + 12t_1t_2 + 12t_2t_3 + 9t_2)$and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 12t_2t_3 + 9t_2 + t_1) + 2$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_2 + 12t_2t_3 + 5t_2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_2t_3 + 5t_2$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_2t_3 + 5t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_2t_3 + 9t_2$ pendent vertices.

Subcase ($\boldsymbol{I\ d}$): $m \equiv 3\ (mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$. We get $p = 4(16t_1t_2t_3 + 16t_1t_2 + 12t_2t_3 + 12t_2)$and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 12t_2t_3 + 12t_2 + t_1) + 2$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2 + 12t_2t_3 + 8t_2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_2t_3 + 8t_2$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_2t_3 + 8t_2$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_2t_3 + 12t_2$ pendent vertices.

Subcase ($\boldsymbol{II}$): $N \equiv 1\ (mod\ 4)$

Then, $N = 4t_2 + 1,\ t_2 > 0$. Assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + t_1 + 4t_2 + 1$ vertices. Then assign the vector $(1,1,1,0)$ to the next $4t_1t_2 + t_1 + 4t_2 + 1$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2 + t_1 + 4t_2 + 1$ vertices. Assign the vector $(1,0,0,0)$ to the next $4t_1t_2 + t_1$ vertices.

Subcase ($\boldsymbol{II\ a}$): $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$.We get $p = 4(16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 12t_2t_3 + t_1 + 3t_2 + 3t_3) + 3$and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 12t_2t_3 + 2t_1 + 3t_2 + 3t_3 + 1) + 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_3 + 12t_2t_3 - t_2 + 3t_3$ pendent vertices. Then assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_3 + 12t_2t_3 - t_2 + 3t_3$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_3 + 12t_2t_3 - t_2 + 3t_3 - 1$ pendent vertices. Assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_3 + 12t_2t_3 + 3t_2 + 3t_3 + 1$ pendent vertices.

Subcase ($\boldsymbol{II\ b}$): $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$.We have $p = 4(16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 12t_2t_3 + 2t_1 + 6t_2 + 3t_3 + 1) + 2$and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 12t_2t_3 + 3t_1 + 6t_2 + 3t_3 + 2)$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 12t_2t_3 + t_1 + 2t_2 + 3t_3 + 1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next$16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 12t_2t_3 + t_1 + 2t_2 + 3t_3$pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 12t_2t_3 + t_1 + 2t_2 + 3t_3$pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 4t_1t_3 + 12t_2t_3 + t_1 + 6t_2 + 3t_3 + 2$ pendent vertices.

Subcase ($\boldsymbol{II\ c}$): $m \equiv 2\ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$.We get $p = 4(16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 12t_2t_3 + 3t_1 + 9t_2 + 3t_3 + 2) + 1$and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 12t_2t_3 + 4t_1 + 9t_2 + 3t_3 + 2) + 3$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 12t_2t_3 + 2t_1 + 5t_2 + 3t_3 + 1$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 12t_2t_3 + 2t_1 + 5t_2 + 3t_3 + 1$ pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 12t_2t_3 + 2t_1 + 5t_2 + 3t_3 + 1$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 4t_1t_3 + 12t_2t_3 + 2t_1 + 9t_2 + 3t_3 + 3$ pendent vertices.

Subcase ($\boldsymbol{II\ d}$): $m \equiv 3\ (mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$.We have $p = 4(16t_1t_2t_3 + 16t_1t_2 + 4t_1t_3 + 12t_2t_3 + 4t_1 + 12t_2 + 3)$and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 4t_1t_3 + 12t_2t_3 + 5t_1 + 12t_2 + 3) + 2$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 12t_2t_3 + 3t_1 + 8t_2 + 2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 12t_2t_3 + 3t_1 + 8t_2 + 2$ pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 12t_2t_3 + 3t_1 + 8t_2 + 2$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 4t_1t_3 + 12t_2t_3 + 3t_1 + 12t_2 + 3$ pendent vertices.

Subcase ($\boldsymbol{III}$): $N \equiv 2\ (mod\ 4)$

Then, $N = 4t_2 + 2,\ t_2 > 0$. Assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + 2t_1 + 4t_2 + 2$ vertices. Then assign the vector $(1,1,1,0)$ to the next $4t_1t_2 + 2t_1 + 4t_2 + 2$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2 + 2t_1 + 4t_2 + 2$vertices. Assign the vector $(1,0,0,0)$ to the next $4t_1t_2 + 2t_1$ vertices.

Subcase ($\boldsymbol{III\ a}$): $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$.We get $p = 4(16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 12t_2t_3 + 2t_1 + 3t_2 + 6t_3 + 1) + 2$and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 12t_2t_3 + 3t_1 + 3t_2 + 6t_3 + 2)$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_3 + 12t_2t_3 - t_2 + 6t_3$ pendent vertices. Then assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_3 + 12t_2t_3 - t_2 + 6t_3 - 1$ pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_3 + 12t_2t_3 - t_2 + 6t_3 - 1$ pendent vertices. Assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_3 + 12t_2t_3 + 3t_2 + 6t_3 + 2$ pendent vertices.

Subcase ($\boldsymbol{III\ b}$): $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$.We have $p = 4(16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 12t_2t_3 + 4t_1 + 6t_2 + 6t_3 + 3)$and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 12t_2t_3 + 5t_1 + 6t_2 + 6t_3 + 3) + 2$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 12t_2t_3 + 2t_1 + 2t_2 + 6t_3 + 1$ pendent vertices and assign the vector

$(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 12t_2t_3 + 2t_1 + 2t_2 + 6t_3 + 1$ pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 12t_2t_3 + 2t_1 + 2t_2 + 6t_3 + 1$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 8t_1t_3 + 12t_2t_3 + 2t_1 + 6t_2 + 6t_3 + 3$pendent vertices.

Subcase $(\boldsymbol{III\ c})$: $m \equiv 2\ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$.We get $p = 4(16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 12t_2t_3 + 6t_1 + 9t_2 + 6t_3 + 4) + 2$and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 12t_2t_3 + 7t_1 + 9t_2 + 6t_3 + 5)$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 12t_2t_3 + 4t_1 + 5t_2 + 6t_3 + 3$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 12t_2t_3 + 4t_1 + 5t_2 + 6t_3 + 2$ pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 12t_2t_3 + 4t_1 + 5t_2 + 6t_3 + 2$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 8t_1t_3 + 12t_2t_3 + 4t_1 + 9t_2 + 6t_3 + 5$pendent vertices.

Subcase $(\boldsymbol{III\ d})$: $m \equiv 3\ (mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$.We get $p = 4(16t_1t_2t_3 + 16t_1t_2 + 8t_1t_3 + 12t_2t_3 + 8t_1 + 12t_2 + 6t_3 + 6)$and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 8t_1t_3 + 12t_2t_3 + 9t_1 + 12t_2 + 6t_3 + 6) + 2$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 12t_2t_3 + 6t_1 + 8t_2 + 6t_3 + 4$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 12t_2t_3 + 6t_1 + 8t_2 + 6t_3 + 4$ pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 12t_2t_3 + 6t_1 + 8t_2 + 6t_3 + 4$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 8t_1t_3 + 12t_2t_3 + 6t_1 + 12t_2 + 6t_3 + 6$pendent vertices.

Subcase $(\boldsymbol{IV})$: $N \equiv 3\ (mod\ 4)$

Then, $N = 4t_2 + 3,\ t_2 \geq 0$. Assign the vector $(1,1,1,1)$ to the first $4t_1t_2 + 3t_1 + 4t_2 + 3$ vertices. Then assign the vector $(1,1,1,0)$ to the next $4t_1t_2 + 3t_1 + 4t_2 + 3$vertices. Also assign the vector $(1,1,0,0)$ to the next $4t_1t_2 + 3t_1 + 4t_2 + 3$vertices. Assign the vector $(1,0,0,0)$ to the next $4t_1t_2 + 3t_1$ vertices.

Subcase $(\boldsymbol{IV\ a})$: $m \equiv 0\ (mod\ 4)$

Then, $m = 4t_3,\ t_3 > 0$.We get $p = 4(16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 12t_2t_3 + 3t_1 + 3t_2 + 9t_3 + 1) + 1$and $q = 4(16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 12t_2t_3 + 4t_1 + 3t_2 + 9t_3 + 2) + 3$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_3 + 12t_2t_3 - t_2 + 9t_3 - 1$ pendent vertices. Then assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 12t_2t_3 - t_2 + 9t_3 - 1$ pendent vertices. Thereafter assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 12t_2t_3 - t_2 + 9t_3 - 1$ pendent vertices. Assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_3 + 12t_2t_3 + 3t_2 + 9t_3 + 3$pendent vertices.

Subcase $(\boldsymbol{IV\ b})$: $m \equiv 1\ (mod\ 4)$

Then, $m = 4t_3 + 1,\ t_3 \geq 0$.We have $p = 4(16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 12t_2t_3 + 6t_1 + 6t_2 + 9t_3 + 4) + 2$and $q = 4(16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 12t_2t_3 + 7t_1 + 6t_2 + 9t_3 + 5)$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 12t_2t_3 + 3t_1 + 2t_2 + 9t_3 + 2$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 12t_2t_3 + 3t_1 + 2t_2 + 9t_3 + 1$ pendent vertices. Then assign the vector $(1,1,0,0)$ to the next$16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 12t_2t_3 + 3t_1 + 2t_2 + 9t_3 + 1$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 4t_1t_2 + 12t_1t_3 + 12t_2t_3 + 3t_1 + 6t_2 + 9t_3 + 5$pendent vertices.

Subcase $(\boldsymbol{IV\ c})$: $m \equiv 2\ (mod\ 4)$

Then, $m = 4t_3 + 2,\ t_3 \geq 0$.We get $p = 4(16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 12t_2t_3 + 9t_1 + 9t_2 + 9t_3 + 6) + 3$and $q = 4(16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 12t_2t_3 + 10t_1 + 9t_2 + 9t_3 + 7) + 1$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 12t_2t_3 + 6t_1 + 5t_2 + 9t_3 + 4$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 12t_2t_3 + 6t_1 + 5t_2 + 9t_3 + 3$ pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 12t_2t_3 + 6t_1 + 5t_2 + 9t_3 + 4$ pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 8t_1t_2 + 12t_1t_3 + 12t_2t_3 + 6t_1 + 9t_2 + 9t_3 + 7$pendent vertices.

Subcase $(\boldsymbol{IV\ d})$: $m \equiv 3\ (mod\ 4)$

Then, $m = 4t_3 + 3,\ t_3 \geq 0$.We have $p = 4(16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 12t_2t_3 + 12t_1 + 9t_2 + 9t_3 + 9)$and $q = 4(16t_1t_2t_3 + 16t_1t_2 + 12t_1t_3 + 12t_2t_3 + 13t_1 + 12t_2 + 9t_3 + 9) + 2$. We assign the vector $(1,1,1,1)$ to the first $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 12t_2t_3 + 9t_1 + 8t_2 + 9t_3 + 6$ pendent vertices and assign the vector $(1,1,1,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 12t_2t_3 + 9t_1 + 8t_2 + 9t_3 + 6$pendent vertices. Then assign the vector $(1,1,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 12t_2t_3 + 9t_1 + 8t_2 + 9t_3 +$

6pendent vertices and assign the vector $(1,0,0,0)$ to the next $16t_1t_2t_3 + 12t_1t_2 + 12t_1t_3 + 12t_2t_3 + 9t_1 + 12t_2 + 9t_3 + 9$pendent vertices.
Hence this vertex labelingφ is a VB $\{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$-cordial labeling of $AGS_{n,N} \odot mK_1$ for $n \geq 2, N \geq 3$ and $m \geq 1$.

CONCLUSION

In this paper, we proved that the corona product of generalized alternate snake graph with m copies of K_1 is a vector basis $\{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$-cordial. The existence of vector basis S-cordial labeling of the corona operation of more families of graphs with m copies of K_1 is an open problem.

REFERENCES

[1] H. Attiya, N. Ahmed and F. Salama,Total edge irregularity strength of star snake graphs, European Journal of Pure and Applied Mathematics, Vol.18Issue 2, (2025), 1-11.
[2] I. Cahit, Cordial Graphs: A weaker version of Graceful and Harmonious Graphs, Ars comb., 23(3), 1987, pp. 201-207.
[3] A. Gallian, A Dynamic Survey of Graph Labeling, Electronic Journal of Combinatorics, Vol. 27, 2024, pp. 1-712.
[4] F. Harary, Graph Theory, Narosa Publishing House, New Delhi, 1988.
[5] I. N. Herstein, Topics in Algebra, John Wiley & Sons, United States of America, 1991.
[6]R. Ponraj and R. Jeya, Vector Basis S-cordial lebeling of graphs, J. Math. Comput. Sci. 15:5 (2025) 1 –13.
[7] R. Ponraj and R. Jeya, Certain VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial thorn graphs, Global Journal of Pure and Applied Mathematics 21(1) (2025) 1–14.
[8] R. Ponraj and R. Jeya, Vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial lebeling of generalized friendship graph, tadpole graph and gear graph, Global Journal of Pure and Applied Mathematics, 21(1) (2025) 81–94.
[9] R. Ponraj and S. Prabhu, Pair mean cordial labeling of hexagonal snake, irregular quadrilateral snake and triple triangular snake, Indian Journal of Natural Sciences 15 Issue 83 (2024) 73075–73081.
[10] R. Ponraj and S. Prabhu, Pair mean cordiality of some snake graphs, Global Journal of Pure and Applied Mathematics 18(1), (2022) 283–295.
[11]A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs, (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris, 1967, pp. 349-355.
[12] S. S. Santhiya, E. Ebin Raja Merly and S. D. Deepa, Heronian mean labeling of graphs, International Mathematical Forum, 12 No. 15, (2017), 705-713.
[13] S. S. Santhiya, E. Ebin Raja Merlyand S. D. Deepa, Heronian mean labeling of double triangular and double quadrilateral snake graphs, International Journal of Mathematics and its Applications, 5 Issue 1-A (2017) 79-88.
[14]S. S. Santhiya, E. Ebin Raja Merlyand S. D. Deepa, Heronian mean labeling of triple triangular and triple quadrilateral snake graphs, International Journal of applied Mathematical Sciences, 9 No.2, (2016), 177-186.
[15]F. Salama,Computing the total edge irregularity strength for heptagonal snake graph and related graphs, Soft Computing, 26(1), (2022), 155-164.
[16]F. Salama,Computing the total edge irregularity strength for quintet snake graph and related graphs, Journal of Discrete Mathematical Sciences and Cryptography, 24(8), (2021), 2491-2504.