International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 14S, 2025
https://www.theaspd.com/ijes.php

On The Lattie Of Subgroup Of The Group Of Upper Triangular Matrices Of Matrix Group

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Abstract
In this paper our main focus is to give the lattice structure $L_u(G)$ of the lattice of subgroups of the groups of the upper triangular matrices of the 2×2 matrices over Z_p under matrix multiplication modulo p, where p is prime. The lattice structure $L_u(G)$ is displayed for $p = 2,3,5$ and 7.The subgroups are singled out by means of Lagrange's theorem, Sylow's theorem, a result of N.BourBaki etc.
Keywords; *Upper triangular matrix, Subgroups, Groups.*

INTRODUCTION

The main focus in this paper is on Lattices of subgroups. The study of subgroup lattices has a quite long history, starting with Richard Dedekind's work [2] in 1877.
The lattice formed by all subgroups of a group G is denoted by $L_u(G)$.
In 2015, D. Jebaraj Thiraviam has given the structure of subgroup lattice of the groups of 2×2 matrices over Z_p having determined value 1 under composition of function matrix multiplication modulo p, where p is one of the prime numbers 2, 3, 5 and 7 and studied various lattice identifies satisfied by it. This motivated us to investigate the lattice structure $L(G)$ of the lattice of subgroups of the upper triangular matrices of the group G of 2×2 matrices over Z_p when $p = 2,3,5$ and 7.
In this paper, we give the lattice structures of the above.

PRELIMINARIES

Definition 1.1.
A partial order on a non-empty set P is a binary relation $\leq$ on P that is reflexive, antisymmetric and transitive.
The pair $(P, \leq)$ is called a partially ordered set or $Poset$. $Poset$ $(P, \leq)$ is totally ordered if every $x, y \in P$ are comparable. That is, if either $x \leq y$ or $y \leq x$.
A non-empty subset S of P is a chain in P if S is totally ordered by $\leq$.
Definition 1.2.
Let $(P, \leq)$ be a poset and let $S \subseteq P$. An upper bound for S is an element $x \in P$ for which $s \leq x \; \forall \, s \in S$.

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The least upper bound of S is called the supremum or join of S.A lower bound of S is an element $x \in P$ for which $x \leq s \forall s \in S$.The greatest upper bound of S is called the infimum or meet of S. A Poset $(P, \leq)$ is called a lattice if every pair $x, y \in P$ has a supremum and an infimum.

Definition 1.3.

In the poset $(p, \leq)$, a covers b or b is covered by a (in notation $a \succ b$ or $b \prec a$) if and only if $b < a$ and, for no $x \in P, b < x < a$.

Definition 1.4.

An element $'a'$ is an atom if $a \succ 0$ and a dual atom, if $a \prec 1$.

Definition 1.5.

A triangular matrix is a special kind of square matrix. A square matrix is called upper triangular if all the entries below the main diagonal are zero.

Definition 1.6.

A subgroup of G of order p^m where $p^m \mid o(G)$ but $p^{m+1} \nmid o(G)$ is called a p-sylow subgroup of G.

Theorem 1.7.[lagrange's Theorem]

If G is a finite group and H is a subgroup of G, then the order of H is a divisor of the order of G.

Theorem 1.8.

If G is a finite group and $a \in G$, then the order of $'a'$ is a divisor of the order of G.

Theorem 1.9.

Let G be a finite group and let p be any prime number that divides the order of G. Then G contains an element of order of p.

Theorem 1.10.[Sylow's Theorem]

If p is a prime number and $p^\alpha | o(G), p^{\alpha+1} \nmid o(G)$, then G has a subgroup of order p^α, called a p-sylow subgroup.

Lemma 1.11.

Let $G = \{(a\ b\ c\ d\ ) : a, b, c, d \in Z_p, ad - bc \neq 0\}, G$is a group under matrix multiplication modulo p. Let G be the set of all upper triangular matrices i.e, $G = \{(a\ b\ 0\ c\ ) : a, b, c \in Z_p , ad - ac \neq 0\}$. Then G is a subgroup of G of order $p(p-1)^2$.

Proof.

The element b can be chosen in p ways.

Since, $a \neq 0$, the number of ways in which a can be chosen is $(p - 1)$.

Also, since $c \neq 0$, the number of ways in which a can be chosen is $(p - 1)$.

Therefore, the total number of ways of choosing a, b and c is

$p(p - 1)(p - 1)$.

That is, $p(p - 1)^2$ ways

Therefore, $o(G) = p(p - 1)^2$.

<u>**Lattice Structure of $L_u(G)$ when $p = 2$.**</u>

$$G = \{(a\ b\ c\ d\ ): a, b, c, d \in Z_2 , ad - bc \neq 0\}$$

$\Rightarrow$ as $o(G) = p(p^2 - 1)(p - 1)$

$= 6$, we have

$$G = \{(1\ 0\ 0\ 1\ ), (0\ 1\ 1\ 1\ ), (1\ 1\ 1\ 0\ ), (0\ 1\ 1\ 0\ ), (1\ 0\ 1\ 1\ ), (1\ 1\ 0\ 1\ )\}$$

Let $G = \{(a\ b\ 0\ c\ ) : a, b, c \in Z_2\}$ be the set of all upper triangular matrices over $Z_2 = \{0,1\}$.

Clearly, $G = \{(1\ 0\ 0\ 1\ ), (1\ 1\ 0\ 1\ )\}$ is a group under matrix multiplication modulo 2 of order 2.

$$G = \{e, (1\ 1\ 0\ 1\ )\}, where\ e = (1\ 0\ 0\ 1\ )$$

The diagram of $L_u(G)$ is as shown in the following figure.

<u>Lattice Structure of $L_u(G)$ when $p = 3$.</u>

Now, $G = \{(a\ b\ c\ d\ ) : a, b, c, d \in Z_3 , ad - bc \neq 0\}$ is a group under multiplication modulo 3 of order 48,

$$since, o(G) = p(p^2 - 1)(p - 1)$$
$$= 3 \times 8 \times 2$$
$$= 48$$

Let $G = \{(a\ b\ 0\ c\ ) : a, b, c \in Z_3\}$ be the set of all upper triangular matrices,

As $o(G) = p(p-1)^2 = 3(3-1)^2 = 12$,

G
$= \{(1\ 0\ 0\ 1\ ), (2\ 0\ 0\ 2\ ), (2\ 0\ 0\ 1\ ), (1\ 0\ 0\ 2\ ), (1\ 1\ 0\ 1\ ), (2\ 2\ 0\ 2\ ), \quad (1\ 1\ 0\ 2\ ), (1\ 2\ 0\ 1\ ), (2\ 1\ 0\ 1\ ), (2\ 2\ 0\ 1$

Clearly, G is a group under matrix multiplication modulo 3 of order 12, which is a subgroup of G.

There are first order, second order, third order and sixth order elements in G.

We arrange the elements according to their orders.

Element of order 1.

$$e = (1\ 0\ 0\ 1\ )$$

Elements of order 2:

$$(1\ 0\ 0\ 2\ ), (1\ 1\ 0\ 2\ ), (1\ 2\ 0\ 2\ ), (2\ 0\ 0\ 1\ ), (2\ 0\ 0\ 2\ ), (2\ 1\ 0\ 1\ ), (2\ 2\ 0\ 1\ )$$

Elements of order 3:

$$(1\ 1\ 0\ 1\ ), (1\ 2\ 0\ 1\ )$$

Elements of order 6:

$$(2\ 1\ 0\ 2\ ), (2\ 2\ 0\ 2\ )$$

Now, we arrange the subgroups of G according to their orders. The two element subgroups of G are,

$H_1 = \{(1\ 0\ 0\ 1\ ), (1\ 0\ 0\ 2\ )\}, H_2 = \{(1\ 0\ 0\ 1\ ), (1\ 1\ 0\ 2\ )\}, H_3 = \{(1\ 0\ 0\ 1\ ), (1\ 2\ 0\ 2\ )\}$
$H_4 = \{(1\ 0\ 0\ 1\ ), (2\ 0\ 0\ 1\ )\}, H_5 = \{(1\ 0\ 0\ 1\ ), (2\ 0\ 0\ 2\ )\}, H_6 = \{(1\ 0\ 0\ 1\ ), (2\ 1\ 0\ 1\ )\},$
$H_7 = \{(1\ 0\ 0\ 1\ ), (2\ 2\ 0\ 1\ )\}$

The three-element Subgroup of G is,

$$K_1 = \{(1\ 0\ 0\ 1\ ), (1\ 1\ 0\ 1\ ), (1\ 2\ 0\ 1\ )\}$$

The four-element subgroups of G are,

$$L_1 = \{\ e, (1\ 0\ 0\ 2\ ), (2\ 0\ 0\ 1\ ), (2\ 0\ 0\ 2\ )\}$$
$$L_2 = \{\ e, (1\ 1\ 0\ 2\ ), (2\ 0\ 0\ 2\ ), (2\ 2\ 0\ 1\ )\}$$
$$L_3 = \{\ e, (1\ 2\ 0\ 2\ ), (2\ 0\ 0\ 2\ ), (2\ 1\ 0\ 1\ )\}$$

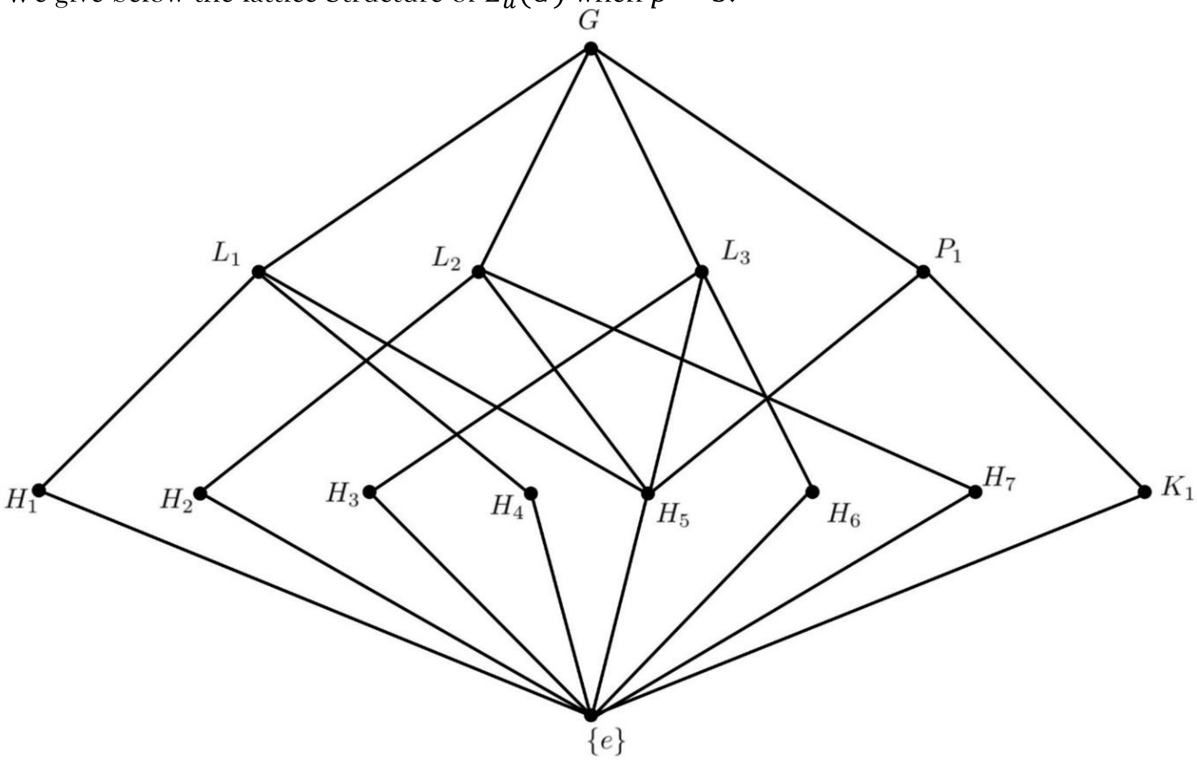
We note that $H_1, H_4, H_5 \subseteq L_1$; $H_2, H_5, H_7 \subseteq L_2$; $H_3, H_5, H_6 \subseteq L_3$

The six-element subgroup of G is,

$$P_1 = \{e, (1\ 1\ 0\ 1\ ), (1\ 2\ 0\ 1\ ), (2\ 0\ 0\ 2\ ), (2\ 1\ 0\ 2\ ), (2\ 2\ 0\ 2\ )\}$$

We note that $K_1 \subseteq P_1$; $H_5 \subseteq P_1$

We give below the lattice Structure of $L_u(G)$ when $p = 3$.



Lattice Structure of $L_u(G)$ when $p = 5$

Let G be the set of all 2×2 non-singular matrices over Z_5. Then G is a group under matrix multiplication modulo 5 and

$$o(G) = p(p^2 - 1)(p - 1)$$
$$= 480$$

Let $G = \{(a\ b\ 0\ c): a, b, c\ \in Z_5\}$ be the set of all upper triangular matrices where

$$Z_5 = \{0,1,2,3,4\}$$

$$we\ have, \quad o(G) = p(p-1)^2$$
$$= 5(5-1)^2$$
$$= 5 \times 16$$
$$= 80$$

Therefore, G is a group under matrix multiplication modulo 5 of order 80, which is a subgroup of G.

We arrange the elements according to their orders.

Element of order 1.

$$e = (1\ 0\ 0\ 1)$$

Elements of order 2.

$(1\ 0\ 0\ 4), (1\ 1\ 0\ 4), (1\ 2\ 0\ 4), (1\ 3\ 0\ 4), (1\ 4\ 0\ 4), (4\ 0\ 0\ 1), (4\ 0\ 0\ 4),$
$(4\ 1\ 0\ 1), (4\ 2\ 0\ 1), (4\ 3\ 0\ 1), (4\ 4\ 0\ 1)$

Elements of order 4.

$(1\ 0\ 0\ 2), (2\ 1\ 0\ 1), (2\ 0\ 0\ 2), (2\ 2\ 0\ 1), (1\ 2\ 0\ 2), (1\ 0\ 0\ 3), (3\ 0\ 0\ 1),$
$(1\ 1\ 0\ 3), (3\ 1\ 0\ 1), (3\ 0\ 0\ 3), (3\ 3\ 0\ 1), (1\ 1\ 0\ 2), (1\ 3\ 0\ 3), (2\ 3\ 0\ 3)$

(3 3 0 2), (2 2 0 3), (2 0 0 3), (3 0 0 2), (3 2 0 2), (2 4 0 4), (4 4 0 2),
(2 2 0 4), (2 0 0 4), (4 0 0 2), (4 2 0 2), (3 4 0 4), (4 4 0 3), (3 3 0 4),
(3 0 0 4), (4 4 0 3), (3 3 0 4), (3 0 0 4), (4 0 0 3), (4 3 0 3), (1 2 0 3),
(2 1 0 3), (3 1 0 2), (3 2 0 1), (2 3 0 1), (1 3 0 2), (2 3 0 4), (3 2 0 4),
(4 2 0 3), (4 3 0 2), (3 4 0 2), (2 4 0 3), (3 1 0 4), (4 1 0 3), (3 4 0 1),
(1 4 0 3), (2 1 0 4), (4 1 0 2), (2 4 0 1), (1 4 0 2), (2 0 0 1).

Elements of order 5.

There are four elements of order 5.

(1 2 0 1), (1 1 0 1), (1 3 0 1), (1 4 0 1)

Elements of order 10.

There are four elements of order 10.

(4 4 0 4), (4 1 0 4), (4 2 0 4), (4 3 0 4).

Elements of order 20.

There are eight elements of order 20.

(2 1 0 2), (3 1 0 3), (3 2 0 3), (2 3 0 2), (2 4 0 2), (3 4 0 3), (3 3 0 3), (2 2 0 2).

Now, we arrange the subgroups of G according to their orders

The two-element subgroups of G.

H_1 to H_{11} are the subgroups of order 2
where

$H_1 = (1\ 0\ 0\ 1), (1\ 0\ 0\ 4),$ $H_2 = (1\ 0\ 0\ 1), (4\ 0\ 0\ 1),$
$H_3 = (1\ 0\ 0\ 1), (1\ 1\ 0\ 4),$ $H_4 = (1\ 0\ 0\ 1), (4\ 1\ 0\ 1),$
$H_5 = (1\ 0\ 0\ 1), (4\ 0\ 0\ 4),$ $H_6 = (1\ 0\ 0\ 1), (1\ 4\ 0\ 4),$
$H_7 = (1\ 0\ 0\ 1), (4\ 4\ 0\ 1),$ $H_8 = (1\ 0\ 0\ 1), (1\ 3\ 0\ 4),$
$H_9 = (1\ 0\ 0\ 1), (4\ 3\ 0\ 1),$ $H_{10} = (1\ 0\ 0\ 1), (1\ 2\ 0\ 4),$
$H_{11} = (1\ 0\ 0\ 1), (4\ 2\ 0\ 1).$

The four-element subgroups of G.

K_1 to K_{31} are the subgroups of order 4, where

$K_1 = (1\ 0\ 0\ 1), (1\ 0\ 0\ 2), (1\ 0\ 0\ 4), (1\ 0\ 0\ 3),$
$K_2 = (1\ 0\ 0\ 1), (2\ 0\ 0\ 3), (4\ 0\ 0\ 4), (3\ 0\ 0\ 2),$
$K_3 = (1\ 0\ 0\ 1), (2\ 0\ 0\ 2), (4\ 0\ 0\ 4), (3\ 0\ 0\ 3),$
$K_4 = (1\ 0\ 0\ 1), (2\ 0\ 0\ 4), (4\ 0\ 0\ 1), (3\ 0\ 0\ 4),$
$K_5 = (1\ 0\ 0\ 1), (3\ 0\ 0\ 1), (4\ 0\ 0\ 1), (2\ 0\ 0\ 1),$
$K_6 = (1\ 0\ 0\ 1), (4\ 0\ 0\ 2), (1\ 0\ 0\ 4), (4\ 0\ 0\ 3),$
$K_7 = (1\ 0\ 0\ 1), (2\ 1\ 0\ 1), (4\ 3\ 0\ 1), (3\ 2\ 0\ 1),$
$K_8 = (1\ 0\ 0\ 1), (2\ 3\ 0\ 3), (4\ 0\ 0\ 4), (3\ 2\ 0\ 2),$
$K_9 = (1\ 0\ 0\ 1), (2\ 2\ 0\ 1), (4\ 1\ 0\ 1), (3\ 4\ 0\ 1),$
$K_{10} = (1\ 0\ 0\ 1), (3\ 3\ 0\ 2), (4\ 0\ 0\ 4), (2\ 2\ 0\ 3),$
$K_{11} = (1\ 0\ 0\ 1), (1\ 2\ 0\ 2), (1\ 1\ 0\ 4), (1\ 4\ 0\ 3),$
$K_{12} = (1\ 0\ 0\ 1), (2\ 4\ 0\ 4), (4\ 4\ 0\ 1), (3\ 2\ 0\ 4),$
$K_{13} = (1\ 0\ 0\ 1), (1\ 0\ 0\ 2), (1\ 0\ 0\ 4), (1\ 0\ 0\ 3),$
$K_{14} = (1\ 0\ 0\ 1), (4\ 0\ 0\ 2), (1\ 4\ 0\ 4), (4\ 2\ 0\ 3),$
$K_{15} = (1\ 0\ 0\ 1), (3\ 1\ 0\ 1), (4\ 4\ 0\ 1), (2\ 3\ 0\ 1),$

$K_{16} = (1\,0\,0\,1), (2\,2\,0\,4), (4\,2\,0\,1), (3\,1\,0\,4)$,
$K_{17} = (1\,0\,0\,1), (3\,3\,0\,1), (4\,2\,0\,1), (2\,4\,0\,1)$,
$K_{18} = (1\,0\,0\,1), (4\,2\,0\,2), (1\,2\,0\,4), (4\,1\,0\,3)$,
$K_{19} = (1\,0\,0\,1), (1\,1\,0\,2), (1\,3\,0\,4), (1\,2\,0\,3)$,
$K_{20} = (1\,0\,0\,1), (3\,4\,0\,4), (4\,3\,0\,1), (2\,3\,0\,4)$,
$K_{21} = (1\,0\,0\,1), (1\,3\,0\,3), (1\,2\,0\,4), (1\,4\,0\,2)$,
$K_{22} = (1\,0\,0\,1), (4\,4\,0\,3), (1\,3\,0\,4), (4\,3\,0\,2)$,
$K_{23} = (1\,0\,0\,1), (3\,3\,0\,4), (4\,1\,0\,1), (2\,1\,0\,4)$,
$K_{24} = (1\,0\,0\,1), (4\,3\,0\,3), (1\,1\,0\,4), (4\,1\,0\,2)$,
$K_{25} = (1\,0\,0\,1), (2\,1\,0\,3), (4\,0\,0\,4), (3\,4\,0\,2)$,
$K_{26} = (1\,0\,0\,1), (2\,4\,0\,3), (4\,0\,0\,4), (3\,1\,0\,2)$,
$K_{27} = (1\,0\,0\,1), (1\,0\,0\,4), (4\,0\,0\,1), (4\,0\,0\,4)$,
$K_{28} = (1\,0\,0\,1), (1\,1\,0\,4), (4\,0\,0\,4), (4\,4\,0\,1)$,
$K_{29} = (1\,0\,0\,1), (4\,1\,0\,1), (4\,0\,0\,4), (1\,4\,0\,4)$,
$K_{30} = (1\,0\,0\,1), (4\,0\,0\,4), (1\,3\,0\,4), (4\,2\,0\,1)$,
$K_{31} = (1\,0\,0\,1), (4\,0\,0\,4), (1\,2\,0\,4), (4\,3\,0\,1)$

$H_1 \subseteq K_1, K_6, K_{27}; H_2 \subseteq K_4, K_5, K_{27}$
$H_3 \subseteq K_{11}, K_{24}, K_{28}; H_4 \subseteq K_9, K_{23}, K_{29};$
$H_5 \subseteq K_2, K_3, K_8, K_{10}, K_{25}, K_{26}, K_{27}, K_{28}, K_{29}, K_{30}, K_{31}$
$H_6 \subseteq K_{13}, K_{14}, K_{29};\ H_7 \subseteq K_{12}, K_{15}, K_{28};$
$H_8 \subseteq K_{19}, K_{22}, K_{30}; H_9 \subseteq K_7, K_{20}, K_{31};$
$H_{10} \subseteq K_{18}, K_{21}, K_{31};\ H_{11} \subseteq K_{16}, K_{17}, K_{30}$

The five-element subgroups of G.
L_1 is the subgroup of order 5 where $L_1 = (1\,0\,0\,1), (1\,2\,0\,1)(1\,1\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1)$.

The eight-element subgroups of G.
M_1 to M_{15} are the subgroups of order 8, where,
$M_1 =$
$(1\,0\,0\,1), (1\,0\,0\,2)(1\,0\,0\,3), (1\,0\,0\,4), (4\,0\,0\,1), (4\,0\,0\,2)(4\,0\,0\,3)(4\,0\,0\,4)$,
$M_2 = (1\,0\,0\,1), (1\,0\,0\,4)(2\,0\,0\,2), (2\,0\,0\,3), (3\,0\,0\,2), (3\,0\,0\,3)(4\,0\,0\,1)(4\,0\,0\,4)$,
$M_3 = (1\,0\,0\,1), (1\,0\,0\,4)(2\,0\,0\,1), (2\,0\,0\,4), (3\,0\,0\,1), (3\,0\,0\,4)(4\,0\,0\,1)(4\,0\,0\,4)$,
$M_4 = (1\,0\,0\,1), (1\,2\,0\,4)(2\,1\,0\,1), (2\,3\,0\,4), (3\,2\,0\,1), (3\,4\,0\,4)(4\,3\,0\,1)(4\,0\,0\,4)$,
$M_5 = (1\,0\,0\,1), (1\,1\,0\,4)(2\,3\,0\,1), (2\,4\,0\,4), (3\,1\,0\,1), (3\,2\,0\,4)(4\,4\,0\,1)(4\,0\,0\,4)$,
$M_6 = (1\,0\,0\,1), (2\,2\,0\,1)(4\,1\,0\,1), (3\,4\,0\,1), (3\,3\,0\,4), (2\,1\,0\,4)(1\,4\,0\,4)(4\,0\,0\,4)$,
$M_7 = (1\,0\,0\,1), (1\,1\,0\,3)(1\,4\,0\,4), (1\,3\,0\,2), (4\,4\,0\,2), (4\,2\,0\,3)(4\,1\,0\,1)(4\,0\,0\,4)$,
$M_8 = (1\,0\,0\,1), (1\,2\,0\,2)(1\,1\,0\,4), (1\,4\,0\,3), (4\,3\,0\,3), (4\,1\,0\,2)(4\,4\,0\,1)(4\,0\,0\,4)$,
$M_9 = (1\,0\,0\,1), (2\,2\,0\,4)(4\,2\,0\,1), (3\,1\,0\,4), (3\,3\,0\,1), (2\,4\,0\,1)(1\,3\,0\,4)(4\,0\,0\,4)$,
$M_{10} = (1\,0\,0\,1), (4\,2\,0\,2)(1\,2\,0\,4), (4\,1\,0\,3), (1\,3\,0\,3), (1\,4\,0\,2)(4\,3\,0\,1)(4\,0\,0\,4)$,
$M_{11} = (1\,0\,0\,1), (1\,3\,0\,4)(1\,1\,0\,2), (1\,2\,0\,3), (4\,4\,0\,3), (4\,3\,0\,2)(4\,2\,0\,1)(4\,0\,0\,4)$,
$M_{12} = (1\,0\,0\,1), (2\,3\,0\,3)(3\,2\,0\,2), (3\,0\,0\,3), (1\,4\,0\,4), (2\,0\,0\,2)(4\,1\,0\,1)(4\,0\,0\,4)$,
$M_{13} = (1\,0\,0\,1), (3\,3\,0\,2)(2\,2\,0\,3), (3\,0\,0\,3), (4\,4\,0\,1), (2\,0\,0\,2)(1\,1\,0\,4)(4\,0\,0\,4)$,

$M_{14} = (1\,0\,0\,1), (4\,2\,0\,1)(3\,4\,0\,2), (3\,0\,0\,3), (1\,3\,0\,4), (2\,0\,0\,2)(2\,1\,0\,3)(4\,0\,0\,4)$,

$M_{15} = (1\,0\,0\,1), (1\,2\,0\,4)(4\,3\,0\,1), (3\,0\,0\,3), (3\,1\,0\,2), (2\,0\,0\,2)(2\,4\,0\,3)(4\,0\,0\,4),$

$$M_1 \supseteq K_1, K_6, K_{27};\ M_2 \supseteq K_2, K_3, K_{27}$$
$$M_3 \supseteq K_4, K_5, K_{27}; M_4 \supseteq K_7, K_{20}, K_{31}$$
$$M_5 \supseteq K_{12}, K_{15}, K_{28};\ M_6 \supseteq K_9, K_{23}, K_{29}$$
$$M_7 \supseteq K_{13}, K_{14}, K_{29}; M_8 \supseteq K_{11}, K_{24}, K_{28}$$
$$M_9 \supseteq K_{16}, K_{17}, K_{30}; M_{10} \supseteq K_{18}, K_{21}, K_{31}$$
$$M_{11} \supseteq K_{19}, K_{22}, K_{30}; M_{12} \supseteq K_3, K_8, K_{29}$$
$$M_{13} \supseteq K_3, K_{10}, K_{28}; M_{14} \supseteq K_3, K_{25}, K_{30}$$
$$M_{15} \supseteq K_3, K_{26}, K_{31}$$

<u>The ten-element subgroups of G.</u>

N_1 to N_3 are the subgroups of order 10, where

$N_1 = (1\,0\,0\,1), (1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (4\,4\,0\,4), (4\,1\,0\,4), (4\,3\,0\,4), (4\,2\,0\,4), (4\,0\,0\,4)$

$N_2 = (1\,0\,0\,1), (1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,0\,0\,4), (1\,2\,0\,4), (1\,1\,0\,4), (1\,3\,0\,4), (1\,4\,0\,4)$

$N_3 = (1\,0\,0\,1), (1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (4\,0\,0\,1), (4\,3\,0\,1), (4\,4\,0\,1), (4\,2\,0\,1), (4\,1\,0\,1)$

where $H_5, L_1 \subseteq N_1$,

$$H_1, H_3, H_6, H_8, H_{10}, L_1 \subseteq N_2;\ H_2, H_4, H_7, H_9, H_{11}, L_1 \subseteq N_3$$

<u>The sixteen-element subgroups of G.</u>

P_1 to P_5 are the subgroups of order 16,where

$P_1 = \{(1\,0\,0\,1), (1\,0\,0\,2), (1\,0\,0\,3), (1\,0\,0\,4), (2\,0\,0\,1), (2\,0\,0\,2), (2\,0\,0\,3), (2\,0\,0\,4), (3\,0\,0\,1), (3\,0\,0\,2), (3\,0\,0\,3), (3\,0\,0\,4)(4\,0\,0\,1), (4\,0\,0\,2), (4\,0\,0\,3), (4\,0\,0\,4)\}$

$P_2 = \{(1\,0\,0\,1), (1\,2\,0\,4), (2\,1\,0\,1), (2\,3\,0\,4), (3\,2\,0\,1), (3\,4\,0\,4), (4\,3\,0\,1), (4\,0\,0\,4), (3\,1\,0\,2), (3\,0\,0\,3), (1\,4\,0\,2), (1\,3\,0\,3)(4\,2\,0\,2), (4\,1\,0\,3), (2\,0\,0\,2), (2\,4\,0\,3)\}$

$P_3 = \{(1\,0\,0\,1), (1\,1\,0\,4), (2\,3\,0\,1), (2\,4\,0\,4), (3\,1\,0\,1), (3\,2\,0\,4), (4\,4\,0\,1), (4\,0\,0\,4), (1\,2\,0\,2), (1\,4\,0\,3), (4\,3\,0\,3), (4\,1\,0\,2)(3\,3\,0\,2), (2\,2\,0\,3), (3\,0\,0\,3), (2\,0\,0\,2)\}$

$P_4 = \{(1\,0\,0\,1), (2\,2\,0\,1), (4\,1\,0\,1), (3\,4\,0\,1), (3\,3\,0\,4), (2\,1\,0\,4), (4\,0\,0\,4), (1\,4\,0\,4), (1\,1\,0\,3), (1\,3\,0\,2), (4\,4\,0\,2), (4\,2\,0\,3)(2\,3\,0\,1), (3\,2\,0\,2), (3\,0\,0\,3), (2\,0\,0\,2)\}$

$P_5 = \{(1\,0\,0\,1), (2\,2\,0\,4), (4\,2\,0\,1), (3\,1\,0\,4), (2\,4\,0\,1), (4\,0\,0\,4), (1\,3\,0\,4), (2\,1\,0\,3), (1\,1\,0\,2), (1\,2\,0\,3), (4\,4\,0\,3), (4\,3\,0\,2)(2\,0\,0\,2), (3\,0\,0\,3), (3\,4\,0\,2), (3\,3\,0\,1)\}$

and $M_1, M_2, M_3 \subseteq P_1;\ M_4, M_{10}, M_{15} \subseteq P_2$

$M_5, M_8, M_{13} \subseteq P_3; M_6, M_7, M_{12} \subseteq P_4$

$M_9, M_{11}, M_{14} \subseteq P_5$

<u>The twenty-element subgroups of G.</u>

Q_1 to Q_7 are the subgroups of order 20, where

Q_1
$= \{(1\,0\,0\,1), (3\,2\,0\,3), (4\,2\,0\,4), (2\,4\,0\,2), (1\,1\,0\,1), (3\,0\,0\,3), (4\,1\,0\,4), (2\,1\,0\,2), (1\,2\,0\,1), (3\,3\,0\,3),$
$(4\,0\,0\,4), (2\,3\,0\,2)(1\,3\,0\,1), (4\,4\,0\,4), (3\,1\,0\,3), (2\,0\,0\,2), (1\,4\,0\,1)(3\,4\,0\,3), (4\,3\,0\,4), (2\,2\,0\,2)\}$
Q_2
$= \{(1\,0\,0\,1), (1\,2\,0\,1), (1\,1\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,0\,0\,2), (1\,2\,0\,2), (1\,1\,0\,2), (1\,3\,0\,2), (1\,4\,0\,2),$
$(1\,0\,0\,3), (1\,1\,0\,3)(1\,2\,0\,3), (1\,3\,0\,3), (1\,4\,0\,3), (1\,0\,0\,4), (1\,3\,0\,4)(1\,1\,0\,4), (1\,4\,0\,4), (1\,2\,0\,4)\}$
Q_3
$= \{(1\,0\,0\,1), (1\,2\,0\,1), (1\,1\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (2\,0\,0\,3), (2\,4\,0\,3), (2\,2\,0\,3), (2\,1\,0\,3), (2\,3\,0\,3),$
$(4\,0\,0\,4), (4\,3\,0\,4)(4\,4\,0\,4), (4\,2\,0\,4), (4\,1\,0\,4), (3\,0\,0\,2), (3\,1\,0\,2)(3\,3\,0\,2), (3\,4\,0\,2), (3\,2\,0\,2)\}$
Q_4
$= \{(1\,0\,0\,1), (1\,2\,0\,1), (1\,1\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (4\,0\,0\,2), (4\,3\,0\,2), (4\,4\,0\,2), (4\,2\,0\,2), (4\,1\,0\,2),$
$(1\,0\,0\,4), (1\,2\,0\,4)(1\,1\,0\,4), (1\,3\,0\,4), (1\,4\,0\,4), (4\,0\,0\,3), (4\,3\,0\,3)(4\,4\,0\,3), (4\,2\,0\,3), (4\,1\,0\,3)\}$
Q_5
$= \{(1\,0\,0\,1), (1\,2\,0\,1), (1\,1\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (3\,0\,0\,1), (3\,1\,0\,1), (3\,2\,0\,1), (3\,3\,0\,1), (3\,4\,0\,1),$
$(4\,0\,0\,1), (4\,1\,0\,1)(4\,2\,0\,1), (4\,3\,0\,1), (4\,4\,0\,1), (2\,0\,0\,1), (2\,1\,0\,1)(2\,2\,0\,1), (2\,3\,0\,1), (2\,4\,0\,1)\}$
Q_6
$= \{(1\,0\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (3\,4\,0\,4), (3\,2\,0\,4), (3\,0\,0\,4), (3\,3\,0\,4), (3\,1\,0\,4), (4\,3\,0\,1),$
$(4\,2\,0\,1), (4\,1\,0\,1)(4\,0\,0\,1), (4\,4\,0\,1), (2\,3\,0\,4), (2\,0\,0\,4), (2\,2\,0\,4)(2\,4\,0\,4), (2\,1\,0\,4), (1\,1\,0\,1)\}$
Q_7
$= \{(1\,0\,0\,1), (1\,0\,0\,4), (4\,0\,0\,4), (1\,4\,0\,1), (4\,1\,0\,4), (4\,4\,0\,1), (1\,1\,0\,4), (4\,1\,0\,1), (4\,2\,0\,4), (1\,4\,0\,4),$
$(1\,3\,0\,4), (4\,2\,0\,1)(4\,3\,0\,4), (1\,2\,0\,4), (1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1)(4\,4\,0\,4), (4\,3\,0\,1), (4\,0\,0\,1)\}$

And $N_1, K_3 \subseteq Q_1; N_2, K_1, K_{11}, K_{13}, K_{19}, K_{21} \subseteq Q_2$
$N_1, K_2, K_8, K_{10}, K_{25}, K_{26} \subseteq Q_3; N_2, K_6, K_{14}, K_{18}, K_{22}, K_{24} \subseteq Q_4$
$N_3, K_5, K_7, K_9, K_{15}, K_{17} \subseteq Q_5; N_3, K_4, K_{12}, K_{16}, K_{20}, K_{23} \subseteq Q_6$
$N_1, N_2, N_3, K_{27}, K_{28}, K_{29}, K_{30}, K_{31} \subseteq Q_7$

The fifty element subgroups of G.

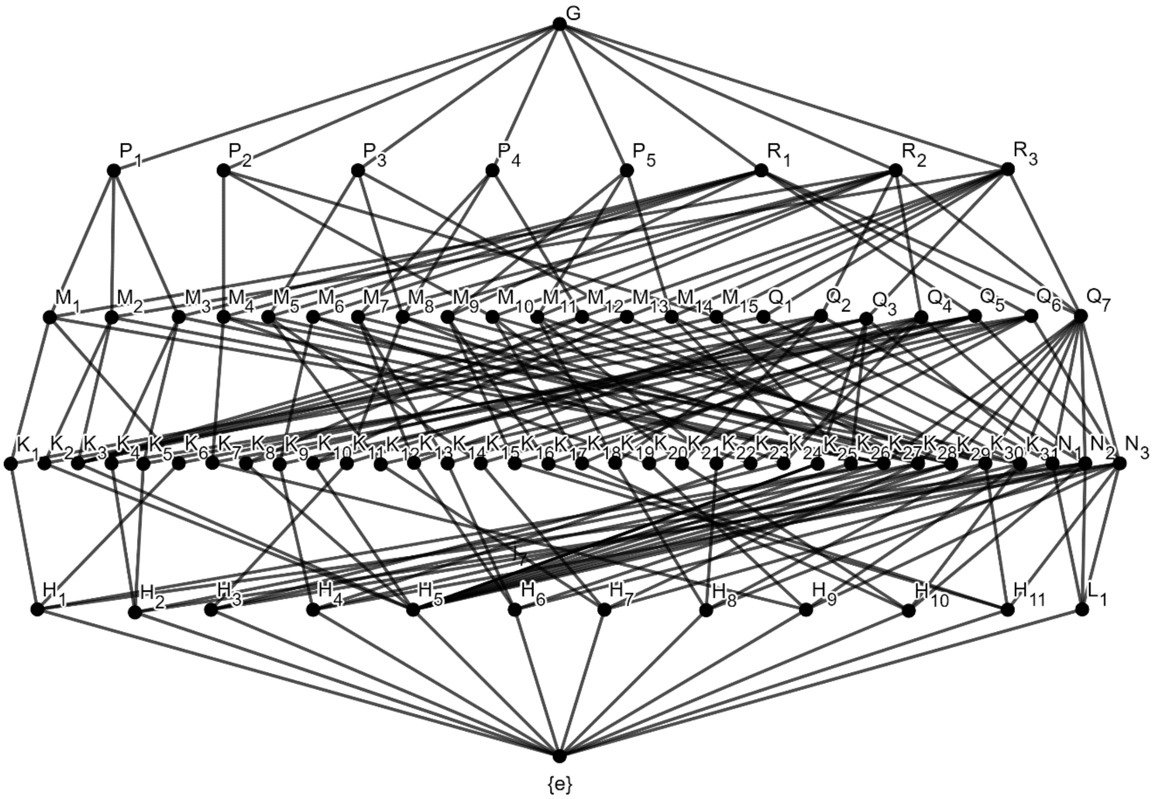
R_1 to R_3 are the subgroups of order 40,where
R_1
$= \{(1\,0\,0\,1), (1\,0\,0\,4), (2\,0\,0\,1), (2\,0\,0\,4), (3\,0\,0\,1), (3\,0\,0\,4), (4\,0\,0\,1), (4\,0\,0\,4), (1\,3\,0\,4), (2\,2\,0\,1),$
$(2\,3\,0\,4), (3\,2\,0\,4)(3\,3\,0\,4), (4\,2\,0\,1), (4\,3\,0\,4), (1\,1\,0\,4), (2\,4\,0\,1)(2\,1\,0\,4), (3\,1\,0\,4), (4\,4\,0\,1), (4\,1\,0\,4), ($
$(2\,1\,0\,1), (3\,4\,0\,4)(4\,1\,0\,1), (4\,4\,0\,4), (1\,2\,0\,4), (4\,3\,0\,1), (4\,2\,0\,4)(1\,1\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1),$
$(1\,2\,0\,1), (3\,2\,0\,1)(3\,4\,0\,1), (3\,1\,0\,1), (3\,3\,0\,1), (2\,3\,0\,1), (2\,4\,0\,4)(2\,2\,0\,4)\}$
R_2
$= \{(1\,0\,0\,1), (1\,3\,0\,4), (1\,1\,0\,2), (1\,2\,0\,3), (4\,4\,0\,3), (4\,3\,0\,2), (4\,0\,0\,4), (4\,2\,0\,1), (1\,0\,0\,2), (1\,2\,0\,1),$
$(1\,1\,0\,4), (4\,3\,0\,4)(4\,4\,0\,1), (4\,2\,0\,2), (4\,0\,0\,3), (1\,1\,0\,1), (1\,0\,0\,3)(1\,3\,0\,2), (4\,0\,0\,2), (4\,3\,0\,3), (4\,3\,0\,1), ($
$(1\,4\,0\,3), (1\,4\,0\,4)(1\,4\,0\,1), (4\,1\,0\,1), (4\,1\,0\,4), (4\,1\,0\,3), (4\,1\,0\,2)(4\,3\,0\,3), (4\,2\,0\,4), (4\,4\,0\,2),$
$(4\,0\,0\,1), (1\,1\,0\,3)(1\,0\,0\,4), (1\,2\,0\,2), (1\,3\,0\,1), (1\,3\,0\,3), (1\,2\,0\,4)(1\,4\,0\,2)\}$
R_3
$= \{(1\,0\,0\,1), (3\,2\,0\,3), (4\,2\,0\,4), (2\,4\,0\,2), (1\,1\,0\,1), (3\,0\,0\,3), (4\,1\,0\,4), (2\,1\,0\,2), (1\,2\,0\,1), (3\,3\,0\,3),$
$(4\,0\,0\,4), (2\,3\,0\,2)(1\,3\,0\,1), (3\,1\,0\,3), (4\,4\,0\,4), (2\,0\,0\,2), (4\,2\,0\,1)(1\,4\,0\,1), (3\,4\,0\,3), (4\,3\,0\,4), (2\,2\,0\,2), ($
$(1\,1\,0\,4), (3\,0\,0\,2)(4\,1\,0\,1), (2\,1\,0\,3), (1\,2\,0\,4), (3\,3\,0\,2), (4\,0\,0\,1)(2\,3\,0\,3), (1\,3\,0\,4), (3\,1\,0\,2),$
$(4\,4\,0\,1), (2\,0\,0\,3)(1\,4\,0\,4), (3\,4\,0\,2), (4\,3\,0\,1), (2\,2\,0\,3), (1\,0\,0\,4)(2\,2\,0\,4)\}$

$M_3, M_4, M_5, M_6, M_9, Q_5, Q_6, Q_7 \subseteq R_1$
$M_1, M_7, M_8, M_{10}, M_{11}, Q_2, Q_4, Q_7 \subseteq R_2$

International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 14S, 2025
https://www.theaspd.com/ijes.php

$$M_2, M_{12}, M_{13}, M_{14}, M_{15}, Q_1, Q_3, Q_7 \subseteq R_3$$

We give below the lattice structure of $L_u(G)$ when $p = 5$.



Lattice Structure of $L_u(G)$ when $p = 7$

Let G be the set of all 2×2 non-singular matrices over Z_7. Then G is a group under matrix multiplication modulo 7 and

$$o(G) = p(p^2 - 1)(p - 1)$$
$$= 2016$$

Let $G = \{(a\ b\ 0\ c): a, b, c \in Z_7\}$ be the set of all upper triangular matrices where

$$Z_7 = \{0,1,2,3,4,5,6\}$$

$$we\ have, \quad o(G) = p(p-1)^2$$
$$= 7(7-1)^2$$
$$= 7 \times 36$$
$$= 252$$

Therefore, G is a group under matrix multiplication modulo 7 of order 252, which is a subgroup of G.

We arrange the elements according to their orders.

Element of order 1.

$$e = (1\ 0\ 0\ 1)$$

Elements of order 2-(15 elements)

$(1\ 0\ 0\ 6), (6\ 0\ 0\ 1), (1\ 1\ 0\ 6), (6\ 1\ 0\ 1), (6\ 6\ 0\ 1), (1\ 6\ 0\ 6), (6\ 0\ 0\ 6), (1\ 2\ 0\ 6), (6\ 2\ 0\ 1), (1\ 3\ 0\ 6), (6\ 3\ 0\ 1), (1\ 4\ 0\ 6), (6\ 4\ 0\ 1), (1\ 5\ 0\ 6), (6\ 5\ 0\ 1)$

Elements of order 3-(44 elements)

(1 0 0 2), (2 0 0 1), (1 1 0 2), (2 1 0 1), (2 0 0 2), (2 2 0 1), (1 2 0 2), (1 0 0 4), (4 0 0 1), (1 1 0 4),
(4 1 0 1), (4 0 0 4), (4 4 0 1), (1 4 0 4), (2 3 0 1), (1 3 0 2), (2 3 0 4), (4 3 0 2), (2 4 0 4), (4 4 0 2),
(2 2 0 4), (2 0 0 4), (4 0 0 2), (4 2 0 2), (1 2 0 4), (2 1 0 4), (4 1 0 2)(2 4 0 1), (1 4 0 2), (4 2 0 1),
(2 5 0 1), (1 5 0 2), (2 6 0 1), (1 6 0 2), (1 3 0 4), (4 3 0 1), (4 5 0 1), (1 5 0 4), (4 6 0 1), (1 6 0 4),
(4 5 0 2), (2 5 0 4), (4 6 0 2), (2 6 0 4).

Elements of order 6-(156 elements)

(1 0 0 3), (3 0 0 1), (1 1 0 3), (3 1 0 1), (3 0 0 3), (3 3 0 1), (1 3 0 3), (1 0 0 5), (5 0 0 1), (1 1 0 5),
(1 2 0 3), (2 1 0 3), (3 1 0 2), (3 2 0 1), (3 2 0 4), (4 2 0 3), (3 4 0 2), (4 3 0 5), (5 3 0 4), (4 5 0 3),
(3 4 0 5), (3 5 0 4), (5 4 0 3), (4 5 0 6), (5 4 0 6), (6 4 0 5), (5 6 0 4)(4 6 0 5), (6 5 0 4), (2 3 0 3),
(3 3 0 2), (2 2 0 3), (2 0 0 3), (3 0 0 2), (3 2 0 2), (2 5 0 5), (5 5 0 2), (2 2 0 5), (2 0 0 5), (5 0 0 2),

(5 2 0 2), (2 6 0 6), (6 6 0 2), (2 2 0 6), (2 0 0 6), (6 0 0 2), (6 2 0 2), (3 4 0 4), (4 4 0 3), (3 3 0 4),
(2 4 0 3), (3 0 0 4), (4 0 0 3), (4 3 0 3), (3 5 0 5), (5 5 0 3), (3 3 0 5), (3 0 0 5), (5 0 0 3), (5 3 0 3),
(3 6 0 6), (6 6 0 3), (3 3 0 6), (3 0 0 6), (6 0 0 3), (6 3 0 3), (4 5 0 5)(4 4 0 5), (5 5 0 4), (4 0 0 5),
(5 0 0 4), (5 4 0 4), (4 6 0 6), (6 6 0 4), (4 4 0 6), (4 0 0 6), (6 0 0 4), (6 4 0 4), (6 5 0 5), (6 6 0 5),
(5 5 0 6), (5 0 0 6), (6 0 0 5), (6 5 0 2), (2 3 0 5), (3 2 0 5), (5 2 0 3), (3 5 0 2), (2 5 0 3), (5 3 0 2),
(2 3 0 6), (3 2 0 6), (6 2 0 3), (3 6 0 2), (2 6 0 3), (6 3 0 2), (3 4 0 6), (4 3 0 6), (6 3 0 4), (4 6 0 3),
(3 6 0 4), (6 4 0 3), (1 2 0 5), (2 1 0 5), (5 1 0 2), (5 2 0 1), (2 1 0 6)(6 1 0 2), (3 1 0 4), (4 1 0 3),
(3 4 0 1), (1 4 0 3), (1 3 0 5), (3 1 0 5), (5 1 0 3), (3 5 0 1), (1 5 0 3), (5 3 0 1), (3 1 0 6), (6 1 0 3),

(3 6 0 1), (1 6 0 3), (1 4 0 5), (4 1 0 5), (5 1 0 4), (5 4 0 1), (4 1 0 6), (6 1 0 4), (2 4 0 5), (4 2 0 5),
(5 2 0 4), (5 4 0 2), (2 4 0 6), (4 2 0 6), (6 2 0 4), (6 4 0 2), (5 3 0 6), (6 3 0 5), (5 6 0 3), (3 6 0 5),
(6 5 0 3), (5 1 0 6), (6 1 0 5), (5 6 0 1), (1 6 0 5), (2 5 0 6), (5 2 0 6)(6 2 0 5), (5 6 0 2), (2 6 0 5),
(5 1 0 1), (5 0 0 5), (5 5 0 1), (1 5 0 5), (2 4 0 3), (6 2 0 5)

Elements of order 7-(6 elements)

(1 1 0 1), (1 2 0 1), (1 3 0 1), (1 4 0 1), (1 5 0 1), (1 6 0 1)

Elements of order 14- (6 elements)

(6 6 0 6), (6 1 0 6), (6 2 0 6), (6 3 0 6), (6 4 0 6), (4 5 0 6)

Elements of order 21-(12 elements)

(2 2 0 2), (2 1 0 2), (4 4 0 4), (2 3 0 2), (4 2 0 4), (2 4 0 2)(2 6 0 2), (4 5 0 4), (4 6 0 4), (2 5 0 2), (4 3 0 4),

Elements of order 42-(12 elements)

(3 3 0 3), (3 1 0 3), (5 1 0 5), (3 2 0 3), (5 0 0 5), (5 3 0 5)(3 5 0 3), (3 4 0 3), (3 6 0 3), (5 6 0 5), (5 4 0 5),

Subgroups of different orders are as given below.

<u>The two-element subgroups of G.</u>

H_1 to H_{15} are the subgroups of order 2

$H_1 =$ (1 0 0 1), (1 0 0 6), $H_2 =$ (1 0 0 1), (6 0 0 1),
$H_3 =$ (1 0 0 1), (1 1 0 6), $H_4 =$ (1 0 0 1), (6 1 0 1),
$H_5 =$ (1 0 0 1), (6 6 0 1), $H_6 =$ (1 0 0 1), (1 6 0 6),
$H_7 =$ (1 0 0 1), (6 0 0 6), $H_8 =$ (1 0 0 1), (1 2 0 6),
$H_9 =$ (1 0 0 1), (6 2 0 1), $H_{10} =$ (1 0 0 1), (1 3 0 6),
$H_{11} =$ (1 0 0 1), (6 3 0 1), $H_{12} =$ (1 0 0 1), (1 4 0 6).
$H_{13} =$ (1 0 0 1), (6 4 0 1), $H_{14} =$ (1 0 0 1), (1 5 0 6).
$H_{15} =$ (1 0 0 1), (6 5 0 1).

<u>The three-element subgroups of G.</u>

K_1 to K_{22} are the subgroups of order 3, where
$K_1 = \{(1\,0\,0\,1), (1\,0\,0\,2), (1\,0\,0\,4)\}$,
$K_2 = \{(1\,0\,0\,1), (2\,0\,0\,1), (4\,0\,0\,1)\}$,
$K_3 = \{(1\,0\,0\,1), (1\,1\,0\,2), (1\,3\,0\,4)\}$,
$K_4 = \{(1\,0\,0\,1), (2\,1\,0\,1), (4\,3\,0\,1)\}$,
$K_5 = \{(1\,0\,0\,1), (2\,0\,0\,2), (4\,0\,0\,4)\}$,
$K_6 = \{(1\,0\,0\,1), (2\,2\,0\,1), (4\,6\,0\,1)\}$,
$K_7 = \{(1\,0\,0\,1), (1\,2\,0\,2), (1\,6\,0\,4)\}$,
$K_8 = \{(1\,0\,0\,1), (1\,1\,0\,4), (1\,5\,0\,2)\}$,
$K_9 = \{(1\,0\,0\,1), (4\,1\,0\,1), (2\,5\,0\,1)\}$,
$K_{10} = \{(1\,0\,0\,1), (4\,4\,0\,1), (2\,6\,0\,1)\}$,
$K_{11} = \{(1\,0\,0\,1), (1\,4\,0\,4), (1\,6\,0\,2)\}$,
$K_{12} = \{(1\,0\,0\,1), (2\,3\,0\,1), (4\,2\,0\,1)\}$,
$K_{13} = \{(1\,0\,0\,1), (1\,3\,0\,2), (1\,2\,0\,4)\}$,
$K_{14} = \{(1\,0\,0\,1), (2\,3\,0\,4), (4\,4\,0\,2)\}$,
$K_{15} = \{(1\,0\,0\,1), (2\,0\,0\,4), (4\,0\,0\,2)\}$,
$K_{16} = \{(1\,0\,0\,1), (2\,4\,0\,4), (4\,3\,0\,2)\}$,
$K_{17} = \{(1\,0\,0\,1), (2\,2\,0\,4), (4\,5\,0\,2)\}$,
$K_{18} = \{(1\,0\,0\,1), (4\,2\,0\,2), (2\,5\,0\,4)\}$,
$K_{19} = \{(1\,0\,0\,1), (2\,1\,0\,4), (4\,6\,0\,2)\}$,
$K_{20} = \{(1\,0\,0\,1), (4\,1\,0\,2), (2\,6\,0\,4)\}$,
$K_{21} = \{(1\,0\,0\,1), (2\,4\,0\,1), (4\,5\,0\,1)\}$,
$K_{22} = \{(1\,0\,0\,1), (1\,4\,0\,2), (1\,5\,0\,4)\}$,

<u>The four-element subgroups of G.</u>

L_1 to L_7 are the subgroups of order 4, where
$L_1 = \{(1\,0\,0\,1), (1\,0\,0\,6), (6\,0\,0\,1), (6\,0\,0\,6)\}$,
$L_2 = \{(1\,0\,0\,1), (1\,1\,0\,6), (6\,0\,0\,6), (6\,6\,0\,1)\}$,
$L_3 = \{(1\,0\,0\,1), (1\,6\,0\,6), (6\,0\,0\,6), (6\,1\,0\,1)\}$,
$L_4 = \{(1\,0\,0\,1), (1\,2\,0\,6), (6\,0\,0\,6), (6\,5\,0\,1)\}$,
$L_5 = \{(1\,0\,0\,1), (6\,2\,0\,1), (6\,0\,0\,6), (1\,5\,0\,6)\}$,
$L_6 = \{(1\,0\,0\,1), (1\,3\,0\,6), (6\,0\,0\,6), (6\,4\,0\,1)\}$,
$L_7 = \{(1\,0\,0\,1), (6\,3\,0\,1), (6\,0\,0\,6), (1\,4\,0\,6)\}$.

$L_1 \supseteq H_1, H_2, H_7$; $L_2 \supseteq H_3, H_5, H_7$
$L_3 \supseteq H_4, H_6, H_7$; $L_4 \supseteq H_7, H_8, H_{15}$
$L_5 \supseteq H_7, H_9, H_{14}$; $L_6 \supseteq H_7, H_{10}, H_{13}$
$L_7 \supseteq H_7, H_{11}, H_{12}$

<u>The six-element subgroups of G.</u>

M_1 to M_{78} are the subgroups of order 6, where,
$M_1 = \{(1\,0\,0\,1), (1\,0\,0\,2)(1\,0\,0\,3), (1\,0\,0\,4), (1\,0\,0\,5), (1\,0\,0\,6)\}$,
$M_2 = \{(1\,0\,0\,1), (3\,0\,0\,1)(2\,0\,0\,1), (4\,0\,0\,1), (5\,0\,0\,1)(6\,0\,0\,1)\}$,
$M_3 = \{(1\,0\,0\,1), (1\,1\,0\,3)(1\,4\,0\,2), (1\,6\,0\,6), (1\,5\,0\,4), (1\,2\,0\,5)\}$,
$M_4 = \{(1\,0\,0\,1), (3\,1\,0\,1)(2\,4\,0\,1), (6\,6\,0\,1), (4\,5\,0\,1), (5\,2\,0\,1)\}$,
$M_5 = \{(1\,0\,0\,1), (3\,0\,0\,3)(2\,0\,0\,2), (4\,0\,0\,4), (5\,0\,0\,5), (6\,0\,0\,6)\}$,

$M_6 = \{(1\,0\,0\,1), (3\,3\,0\,1)(2\,5\,0\,1), (6\,4\,0\,1), (4\,1\,0\,1), (5\,6\,0\,1)\},$
$M_7 = \{(1\,0\,0\,1), (1\,3\,0\,3)(1\,5\,0\,2), (1\,4\,0\,6), (1\,1\,0\,4), (1\,6\,0\,5)\},$
$M_8 = \{(1\,0\,0\,1), (1\,1\,0\,5)(1\,6\,0\,4), (1\,3\,0\,6), (1\,2\,0\,2), (1\,4\,0\,3)\},$
$M_9 = \{(1\,0\,0\,1), (5\,1\,0\,1)(4\,6\,0\,1), (6\,3\,0\,1), (2\,2\,0\,1), (3\,4\,0\,1)\},$
$M_{10} = \{(1\,0\,0\,1), (5\,5\,0\,1)(4\,2\,0\,1), (6\,1\,0\,1), (2\,3\,0\,1), (3\,6\,0\,1)\},$
$M_{11} = \{(1\,0\,0\,1), (1\,5\,0\,5)(1\,2\,0\,4), (1\,1\,0\,6), (1\,3\,0\,2), (1\,6\,0\,3)\},$
$M_{12} = \{(1\,0\,0\,1), (1\,2\,0\,3)(1\,1\,0\,2), (1\,5\,0\,6), (1\,3\,0\,4), (1\,4\,0\,5)\},$
$M_{13} = \{(1\,0\,0\,1), (2\,1\,0\,3)(4\,5\,0\,2), (1\,5\,0\,6), (2\,2\,0\,4), (4\,1\,0\,5)\},$
$M_{14} = \{(1\,0\,0\,1), (3\,1\,0\,2)(2\,5\,0\,4), (6\,5\,0\,1), (4\,2\,0\,2), (5\,1\,0\,4)\},$
$M_{15} = \{(1\,0\,0\,1), (3\,2\,0\,1)(2\,1\,0\,1), (6\,5\,0\,1), (4\,3\,0\,1), (5\,4\,0\,1)\},$
$M_{16} = \{(1\,0\,0\,1), (3\,2\,0\,4)(2\,0\,0\,2), (6\,4\,0\,1), (4\,0\,0\,4), (5\,1\,0\,2)\},$
$M_{17} = \{(1\,0\,0\,1), (4\,2\,0\,3)(2\,0\,0\,2), (1\,4\,0\,6), (4\,0\,0\,4)(2\,1\,0\,5)\},$
$M_{18} = \{(1\,0\,0\,1), (3\,4\,0\,2)(2\,6\,0\,4), (6\,6\,0\,1), (4\,1\,0\,2), (5\,4\,0\,4)\},$
$M_{19} = \{(1\,0\,0\,1), (2\,4\,0\,3)(4\,6\,0\,2), (1\,6\,0\,6), (2\,1\,0\,4), (4\,4\,0\,5)\},$
$M_{20} = \{(1\,0\,0\,1), (3\,4\,0\,5)(2\,4\,0\,4), (6\,0\,0\,6), (4\,3\,0\,2), (5\,3\,0\,3)\},$
$M_{21} = \{(1\,0\,0\,1), (4\,3\,0\,5)(2\,6\,0\,4), (1\,1\,0\,6), (4\,1\,0\,2), (2\,3\,0\,3)\},$
$M_{22} = \{(1\,0\,0\,1), (5\,3\,0\,4)(4\,6\,0\,2), (6\,1\,0\,1), (2\,1\,0\,4)(3\,3\,0\,2)\},$
$M_{23} = \{(1\,0\,0\,1), (4\,5\,0\,3)(2\,0\,0\,2), (1\,3\,0\,6), (4\,0\,0\,4), (2\,6\,0\,5)\},$
$M_{24} = \{(1\,0\,0\,1), (3\,5\,0\,4)(2\,0\,0\,2), (6\,3\,0\,1), (4\,0\,0\,4), (5\,6\,0\,2)\},$
$M_{25} = \{(1\,0\,0\,1), (5\,4\,0\,3)(4\,4\,0\,2), (6\,0\,0\,6), (2\,3\,0\,4), (3\,3\,0\,5)\},$
$M_{26} = \{(1\,0\,0\,1), (4\,5\,0\,6)(2\,1\,0\,1), (1\,2\,0\,6), (4\,3\,0\,1), (2\,3\,0\,6)\},$
$M_{27} = \{(1\,0\,0\,1), (5\,4\,0\,6)(4\,2\,0\,1), (6\,0\,0\,6), (2\,3\,0\,1)(3\,5\,0\,6)\},$
$M_{28} = \{(1\,0\,0\,1), (6\,4\,0\,5)(1\,2\,0\,4), (6\,0\,0\,6), (1\,3\,0\,2), (6\,5\,0\,3)\},$
$M_{29} = \{(1\,0\,0\,1), (5\,6\,0\,4)(4\,5\,0\,2), (6\,2\,0\,1), (2\,2\,0\,4), (3\,6\,0\,2)\},$
$M_{30} = \{(1\,0\,0\,1), (4\,6\,0\,5)(2\,5\,0\,4), (1\,2\,0\,6), (4\,2\,0\,2), (2\,6\,0\,3)\},$
$M_{31} = \{(1\,0\,0\,1), (6\,5\,0\,4)(1\,1\,0\,2), (6\,2\,0\,1), (1\,3\,0\,4), (6\,3\,0\,3)\},$
$M_{32} = \{(1\,0\,0\,1), (2\,2\,0\,3)(4\,3\,0\,2), (1\,3\,0\,6), (2\,4\,0\,4)(4\,2\,0\,5)\},$
$M_{33} = \{(1\,0\,0\,1), (2\,0\,0\,3)(4\,0\,0\,2), (1\,0\,0\,6), (2\,0\,0\,4), (4\,0\,0\,5)\},$
$M_{34} = \{(1\,0\,0\,1), (3\,0\,0\,2)(2\,0\,0\,4), (6\,0\,0\,1), (4\,0\,0\,2), (5\,0\,0\,4)\},$
$M_{35} = \{(1\,0\,0\,1), (3\,2\,0\,2)(2\,3\,0\,4), (6\,3\,0\,1), (4\,4\,0\,2), (5\,2\,0\,4)\},$
$M_{36} = \{(1\,0\,0\,1), (2\,5\,0\,5)(4\,0\,0\,4), (1\,6\,0\,6), (2\,0\,0\,2), (4\,3\,0\,3)\},$
$M_{37} = \{(1\,0\,0\,1), (3\,0\,0\,1)(2\,0\,0\,1), (4\,0\,0\,1), (5\,0\,0\,1)(6\,0\,0\,1)\},$
$M_{38} = \{(1\,0\,0\,1), (2\,2\,0\,5)(4\,0\,0\,4), (1\,1\,0\,6), (2\,0\,0\,2), (4\,4\,0\,3)\},$
$M_{39} = \{(1\,0\,0\,1), (2\,0\,0\,5)(4\,0\,0\,4), (1\,0\,0\,6), (2\,0\,0\,2), (4\,0\,0\,3)\},$
$M_{40} = \{(1\,0\,0\,1), (5\,0\,0\,2)(4\,0\,0\,4), (6\,0\,0\,1), (2\,0\,0\,2), (3\,0\,0\,4)\},$
$M_{41} = \{(1\,0\,0\,1), (5\,2\,0\,2)(4\,0\,0\,4), (6\,1\,0\,1), (2\,0\,0\,2), (3\,4\,0\,4)\},$
$M_{42} = \{(1\,0\,0\,1), (2\,6\,0\,6)(4\,6\,0\,1), (1\,4\,0\,6), (2\,2\,0\,1)(4\,3\,0\,6)\},$
$M_{43} = \{(1\,0\,0\,1), (6\,6\,0\,2)(1\,6\,0\,4), (6\,4\,0\,1), (1\,2\,0\,2), (6\,3\,0\,4)\},$
$M_{44} = \{(1\,0\,0\,1), (2\,2\,0\,6)(4\,2\,0\,1), (1\,6\,0\,6), (2\,3\,0\,1), (4\,1\,0\,6)\},$
$M_{45} = \{(1\,0\,0\,1), (2\,0\,0\,6)(4\,0\,0\,1), (1\,0\,0\,6), (2\,0\,0\,1), (4\,0\,0\,6)\},$
$M_{46} = \{(1\,0\,0\,1), (6\,0\,0\,2)(1\,0\,0\,4), (6\,0\,0\,1), (1\,0\,0\,2), (6\,0\,0\,4)\},$
$M_{47} = \{(1\,0\,0\,1), (6\,2\,0\,2)(1\,2\,0\,4), (6\,6\,0\,1), (1\,3\,0\,2)(6\,1\,0\,4)\},$
$M_{48} = \{(1\,0\,0\,1), (3\,5\,0\,5)(2\,5\,0\,4), (6\,0\,0\,6), (4\,2\,0\,2), (5\,2\,0\,3)\},$
$M_{49} = \{(1\,0\,0\,1), (5\,5\,0\,3)(4\,5\,0\,2), (6\,0\,0\,6), (2\,2\,0\,4), (3\,2\,0\,5)\},$

$M_{50} = \{(1\,0\,0\,1), (3\,0\,0\,5)(2\,0\,0\,4), (6\,0\,0\,6), (4\,0\,0\,2), (5\,0\,0\,3)\},$
$M_{51} = \{(1\,0\,0\,1), (3\,6\,0\,6)(2\,5\,0\,1), (6\,0\,0\,6), (4\,1\,0\,1), (5\,2\,0\,6)\},$
$M_{52} = \{(1\,0\,0\,1), (6\,6\,0\,3)(1\,5\,0\,2), (6\,0\,0\,6), (1\,1\,0\,4)(6\,2\,0\,5)\},$
$M_{53} = \{(1\,0\,0\,1), (3\,0\,0\,6)(2\,0\,0\,1), (6\,0\,0\,6), (4\,0\,0\,1), (5\,0\,0\,6)\},$
$M_{54} = \{(1\,0\,0\,1), (6\,0\,0\,3)(1\,0\,0\,2), (6\,0\,0\,6), (1\,0\,0\,4), (6\,0\,0\,5)\},$
$M_{55} = \{(1\,0\,0\,1), (6\,3\,0\,3)(1\,6\,0\,2), (6\,0\,0\,6), (1\,4\,0\,4), (6\,1\,0\,5)\},$
$M_{56} = \{(1\,0\,0\,1), (4\,5\,0\,5)(2\,3\,0\,4), (1\,4\,0\,6), (4\,4\,0\,2), (2\,5\,0\,3)\},$
$M_{57} = \{(1\,0\,0\,1), (4\,6\,0\,6)(2\,4\,0\,1), (1\,1\,0\,6), (4\,5\,0\,1)(2\,5\,0\,6)\},$
$M_{58} = \{(1\,0\,0\,1), (6\,6\,0\,4)(1\,4\,0\,2), (6\,1\,0\,1), (1\,5\,0\,4), (6\,5\,0\,2)\},$
$M_{59} = \{(1\,0\,0\,1), (4\,4\,0\,6)(2\,5\,0\,1), (1\,3\,0\,6), (4\,1\,0\,1), (2\,1\,0\,6)\},$
$M_{60} = \{(1\,0\,0\,1), (6\,4\,0\,4)(1\,5\,0\,2), (6\,3\,0\,1), (1\,1\,0\,4), (6\,1\,0\,2)\},$
$M_{61} = \{(1\,0\,0\,1), (6\,5\,0\,5)(1\,6\,0\,4), (6\,0\,0\,6), (1\,2\,0\,2), (6\,1\,0\,3)\},$
$M_{62} = \{(1\,0\,0\,1), (6\,6\,0\,5)(1\,3\,0\,4), (6\,0\,0\,6), (1\,1\,0\,2)(6\,4\,0\,3)\},$
$M_{63} = \{(1\,0\,0\,1), (5\,5\,0\,6)(4\,6\,0\,1), (6\,0\,0\,6), (2\,2\,0\,1), (3\,1\,0\,6)\},$
$M_{64} = \{(1\,0\,0\,1), (2\,3\,0\,5)(4\,0\,0\,4), (1\,5\,0\,6), (2\,0\,0\,2), (4\,6\,0\,3)\},$
$M_{65} = \{(1\,0\,0\,1), (5\,6\,0\,6)(4\,3\,0\,1), (6\,0\,0\,6), (2\,1\,0\,1), (3\,4\,0\,6)\},$
$M_{66} = \{(1\,0\,0\,1), (3\,5\,0\,2)(2\,4\,0\,4), (6\,4\,0\,1), (4\,3\,0\,2), (5\,5\,0\,4)\},$
$M_{67} = \{(1\,0\,0\,1), (5\,3\,0\,2)(4\,0\,0\,4), (6\,5\,0\,1), (2\,0\,0\,2)(3\,6\,0\,4)\},$
$M_{68} = \{(1\,0\,0\,1), (6\,2\,0\,3)(1\,4\,0\,2), (6\,0\,0\,6), (1\,5\,0\,4), (6\,3\,0\,5)\},$
$M_{69} = \{(1\,0\,0\,1), (5\,1\,0\,6)(4\,4\,0\,1), (6\,0\,0\,6), (2\,6\,0\,1), (3\,3\,0\,6)\},$
$M_{70} = \{(1\,0\,0\,1), (5\,6\,0\,3)(4\,6\,0\,2), (6\,0\,0\,6), (2\,1\,0\,4), (3\,1\,0\,5)\},$
$M_{71} = \{(1\,0\,0\,1), (5\,3\,0\,6)(4\,5\,0\,1), (6\,0\,0\,6), (2\,4\,0\,1), (3\,2\,0\,6)\},$
$M_{72} = \{(1\,0\,0\,1), (6\,4\,0\,2)(1\,4\,0\,4), (6\,5\,0\,1), (1\,6\,0\,2)(6\,2\,0\,4)\},$
$M_{73} = \{(1\,0\,0\,1), (4\,2\,0\,6)(2\,6\,0\,1), (1\,5\,0\,6), (4\,4\,0\,1), (2\,4\,0\,6)\},$
$M_{74} = \{(1\,0\,0\,1), (5\,4\,0\,2)(4\,0\,0\,4), (6\,2\,0\,1), (2\,0\,0\,2), (3\,1\,0\,4)\},$
$M_{75} = \{(1\,0\,0\,1), (3\,6\,0\,5)(2\,6\,0\,4), (6\,0\,0\,6), (4\,1\,0\,2), (5\,1\,0\,3)\},$
$M_{76} = \{(1\,0\,0\,1), (2\,4\,0\,5)(4\,0\,0\,4), (1\,2\,0\,6), (2\,0\,0\,2), (4\,1\,0\,3)\},$
$M_{77} = \{(1\,0\,0\,1), (1\,3\,0\,5)(1\,4\,0\,4), (1\,2\,0\,6), (1\,6\,0\,2), (1\,5\,0\,3)\},$
$M_{78} = \{(1\,0\,0\,1), (3\,5\,0\,1)(2\,6\,0\,1), (6\,2\,0\,1), (4\,4\,0\,1), (5\,3\,0\,1)\},$

$M_1 \supseteq H_1, K_1$; $M_2 \supseteq H_2, K_2$; $M_3 \supseteq H_6, K_{22}$;
$M_4 \supseteq H_5, K_{21}$; $M_5 \supseteq H_7, K_5$; $M_6 \supseteq H_{13}, K_9$;
$M_7 \supseteq H_{12}, K_8$; $M_8 \supseteq H_{10}, K_7$; $M_9 \supseteq H_{11}, K_6$;
$M_{10} \supseteq H_4, K_{12}$; $M_{11} \supseteq H_3, K_{13}$; $M_{12} \supseteq H_{14}, K_3$;
$M_{13} \supseteq H_{14}, K_{17}$; $M_{14} \supseteq H_{15}, K_{18}$; $M_{15} \supseteq H_{15}, K_4$;
$M_{16} \supseteq H_{13}, K_5$; $M_{17} \supseteq H_{12}, K_5$; $M_{18} \supseteq H_5, K_{20}$;
$M_{19} \supseteq H_6, K_{19}$; $M_{20} \supseteq H_7, K_{16}$; $M_{21} \supseteq H_3, K_{20}$;
$M_{22} \supseteq H_4, K_{19}$; $M_{23} \supseteq H_{10}, K_5$; $M_{24} \supseteq H_{11}, K_5$;
$M_{25} \supseteq H_7, K_{14}$; $M_{26} \supseteq H_8, K_4$; $M_{27} \supseteq H_7, K_{12}$;
$M_{28} \supseteq H_7, K_{13}$; $M_{29} \supseteq H_9, K_{17}$; $M_{30} \supseteq H_8, K_{18}$;
$M_{31} \supseteq H_9, K_3$; $M_{32} \supseteq H_{10}, K_{16}$; $M_{33} \supseteq H_1, K_{15}$;
$M_{34} \supseteq H_2, K_{15}$; $M_{35} \supseteq H_{11}, K_{14}$; $M_{36} \supseteq H_6, K_5$;
$M_{37} \supseteq H_5, K_5$; $M_{38} \supseteq H_3, K_5$; $M_{39} \supseteq H_1, K_5$;
$M_{40} \supseteq H_2, K_5$; $M_{41} \supseteq H_4, K_5$; $M_{42} \supseteq H_{12}, K_6$;

$M_{43} \supseteq H_{13}, K_7$; $M_{44} \supseteq H_6, K_{12}$; $M_{45} \supseteq H_1, K_2$;
$M_{46} \supseteq H_2, K_1$; $M_{47} \supseteq H_5, K_{13}$; $M_{48} \supseteq H_7, K_{18}$;
$M_{49} \supseteq H_7, K_{17}$; $M_{50} \supseteq H_7, K_{15}$; $M_{51} \supseteq H_7, K_9$;
$M_{52} \supseteq H_7, K_8$; $M_{53} \supseteq H_7, K_2$; $M_{54} \supseteq H_7, K_1$;
$M_{55} \supseteq H_7, K_{11}$; $M_{56} \supseteq H_{12}, K_{14}$; $M_{57} \supseteq H_3, K_2$;
$M_{58} \supseteq H_4, K_{22}$; $M_{59} \supseteq H_{10}, K_9$; $M_{60} \supseteq H_{11}, K_8$;
$M_{61} \supseteq H_{14}, K_5$; $M_{62} \supseteq H_7, K_3$; $M_{63} \supseteq H_7, K_6$;
$M_{64} \supseteq H_{14}, K_5$; $M_{65} \supseteq H_7, K_4$; $M_{66} \supseteq H_{13}, K_{16}$;
$M_{67} \supseteq H_{15}, K_5$; $M_{68} \supseteq H_7, K_{22}$; $M_{69} \supseteq H_7, K_{10}$;
$M_{58} \supseteq H_4, K_{22}$; $M_{59} \supseteq H_{10}, K_9$; $M_{60} \supseteq H_{11}, K_8$;
$M_{70} \supseteq H_7, K_{19}$; $M_{71} \supseteq H_7, K_{21}$; $M_{72} \supseteq H_5, K_{11}$;
$M_{73} \supseteq H_4, K_{10}$; $M_{74} \supseteq H_9, K_5$; $M_{75} \supseteq H_7, K_{20}$;
$M_{76} \supseteq H_8, K_5$; $M_{77} \supseteq H_8, K_{11}$; $M_{78} \supseteq H_9, K_{10}$

The seven -element subgroup of G -

N_1 is the subgroup of order 7, where,

$$N_1 = \{(1\ 1\ 0\ 1), (1\ 2\ 0\ 1)(1\ 3\ 0\ 1), (1\ 4\ 0\ 1), (1\ 5\ 0\ 1), (1\ 6\ 0\ 1), (1\ 0\ 0\ 1)\}.$$

The nine -element subgroups of G -

P_1 to P_7 are the subgroups of order 9, where,

$$P_1 = \{(1\ 0\ 0\ 2), (1\ 0\ 0\ 4)(2\ 0\ 0\ 2), (2\ 0\ 0\ 4), (4\ 0\ 0\ 2), (4\ 0\ 0\ 4), (2\ 0\ 0\ 1)(4\ 0\ 0\ 1)(1\ 0\ 0\ 1)\},$$

$$P_2 = \{(1\ 1\ 0\ 2), (1\ 3\ 0\ 4)(2\ 0\ 0\ 2), (4\ 0\ 0\ 4), (2\ 2\ 0\ 4), (4\ 5\ 0\ 2), (2\ 6\ 0\ 1)(4\ 4\ 0\ 1)(1\ 0\ 0\ 1)\},$$

$$P_3 = \{(2\ 2\ 0\ 1)(4\ 6\ 0\ 1), (2\ 0\ 0\ 2), (4\ 4\ 0\ 2), (2\ 3\ 0\ 4), (4\ 0\ 0\ 4)(1\ 5\ 0\ 2)(1\ 1\ 0\ 4), (1\ 0\ 0\ 1)\},$$

$$P_4 = \{(1\ 2\ 0\ 2), (1\ 6\ 0\ 4)(2\ 5\ 0\ 1), (2\ 0\ 0\ 2), (2\ 4\ 0\ 4), (4\ 1\ 0\ 1), (4\ 3\ 0\ 2)(4\ 0\ 0\ 4)(1\ 0\ 0\ 1)\},$$

$$P_5 = \{(2\ 1\ 0\ 1), (4\ 3\ 0\ 1)(2\ 0\ 0\ 2), (1\ 4\ 0\ 4), (4\ 2\ 0\ 2), (1\ 6\ 0\ 2), (4\ 0\ 0\ 4)(2\ 5\ 0\ 4)(1\ 0\ 0\ 1)\},$$

$$P_6 = \{(2\ 3\ 0\ 1), (4\ 2\ 0\ 1)(1\ 4\ 0\ 2), (1\ 5\ 0\ 4), (2\ 0\ 0\ 2), (4\ 6\ 0\ 2), (2\ 1\ 0\ 4)(4\ 0\ 0\ 4)(1\ 0\ 0\ 1)\},$$

$$P_7 = \{(1\ 3\ 0\ 2), (1\ 2\ 0\ 4)(4\ 1\ 0\ 2), (2\ 6\ 0\ 4), (4\ 0\ 0\ 4), (2\ 4\ 0\ 1), (4\ 5\ 0\ 1)(2\ 0\ 0\ 2)(1\ 0\ 0\ 1)\}.$$

$P_1 \supseteq K_1, K_2, K_5, K_{15}$; $P_2 \supseteq K_3, K_5, K_{10}, K_{17}$; $P_3 \supseteq K_5, K_6, K_8, K_{14}$
$P_4 \supseteq K_5, K_7, K_9, K_{16}$; $P_5 \supseteq K_4, K_5, K_{11}, K_{18}$; $P_6 \supseteq K_5, K_{12}, K_{19}, K_{22}$
$P_7 \supseteq K_5, K_{13}, K_{20}, K_{21}$

<u>The twelve -element subgroups of G -</u>

Q_1 to Q_{28} are the subgroups of order 12, where,

$Q_1 =$
{(1 0 0 2), (1 0 0 3), (1 0 0 4), (1 0 0 5), (1 0 0 6), (6 0 0 1), (6 0 0 2), (6 0 0 3), (6 0 0 4), (6 0 0 5), (6 0 0

$Q_2 =$
{(1 0 0 6), (3 0 0 1), (2 0 0 1), (6 0 0 1), (4 0 0 1), (5 0 0 1), (2 0 0 6), (3 0 0 6), (4 0 0 6), (5 0 0 6), (6 0 0

$Q_3 =$
{(1 1 0 3), (1 4 0 2), (1 6 0 6), (1 5 0 4), (1 2 0 5), (6 2 0 3), (6 5 0 2), (6 0 0 6), (6 6 0 4), (6 1 0 1), (6 3 0

$Q_4 =$
{(3 1 0 1), (2 4 0 1), (6 6 0 1), (4 5 0 1), (5 2 0 1), (3 2 0 6), (2 5 0 6), (6 0 0 6), (1 1 0 6), (4 6 0 6), (5 3 0

$Q_5 =$
{(3 0 0 3), (2 0 0 2), (4 0 0 4), (5 0 0 5), (6 0 0 6), (2 1 0 5), (6 3 0 1), (4 2 0 3), (5 6 0 2), (1 4 0 6), (3 5 0

$Q_6 =$
{(3 3 0 1), (2 5 0 1), (6 4 0 1), (4 1 0 1), (5 6 0 1), (2 1 0 6), (4 4 0 6), (5 2 0 6), (3 6 0 6), (6 0 0 6), (1 3 0

$Q_7 =$
{(1 3 0 3), (1 5 0 2), (1 4 0 6), (1 1 0 4), (1 6 0 5), (6 1 0 2), (6 6 0 3), (6 0 0 6), (6 4 0 4), (6 2 0 5), (6 3 0

$Q_8 =$
{(1 1 0 5), (1 6 0 4), (1 3 0 6), (1 2 0 2), (1 4 0 3), (6 1 0 3), (6 6 0 2), (6 4 0 1), (6 5 0 5), (6 3 0 4), (6 0 0

$Q_9 =$
{(5 1 0 1), (4 6 0 1), (6 3 0 1), (2 2 0 1), (3 4 0 1), (3 1 0 6), (1 4 0 6), (5 5 0 6), (4 3 0 6), (6 0 0 6), (2 6 0

$Q_{10} =$
{(5 5 0 1), (4 2 0 1), (6 1 0 1), (2 3 0 1), (3 6 0 1), (4 1 0 6), (6 0 0 6), (2 2 0 6), (3 5 0 6), (1 6 0 6), (5 4 0

$Q_{11} =$
{(1 5 0 5), (1 2 0 4), (1 1 0 6), (1 3 0 2), (1 6 0 3), (6 5 0 3), (6 6 0 1), (6 4 0 5), (6 1 0 4), (6 0 0 6), (6 2 0

$Q_{12} =$
{(1 2 0 3), (1 1 0 2), (1 5 0 6), (1 3 0 4), (1 4 0 5), (6 5 0 4), (6 6 0 5), (6 2 0 1), (6 4 0 3), (6 3 0 2), (6 0 0

$Q_{13} =$
{(2 1 0 3), (4 5 0 2), (1 5 0 6), (2 2 0 4), (4 1 0 5), (5 6 0 4), (3 2 0 5), (6 2 0 1), (5 5 0 3), (3 6 0 2), (6 0 0

$Q_{14} =$
{(3 1 0 2), (2 5 0 4), (6 5 0 1), (4 2 0 2), (5 1 0 4), (1 2 0 6), (3 5 0 5), (2 6 0 3), (6 0 0 6), (4 6 0 5), (5 2 0

$Q_{15} =$
{(3 2 0 1), (2 1 0 1), (6 5 0 1), (4 3 0 1), (5 4 0 1), (4 5 0 6), (5 6 0 6), (1 2 0 6), (3 4 0 6), (2 3 0 6), (6 0 0

$Q_{16} =$
{(3 2 0 4), (2 0 0 2), (6 4 0 1), (4 0 0 4), (5 1 0 2), (5 0 0 5), (1 3 0 6), (3 0 0 3), (2 6 0 5), (6 0 0 6), (4 5 0

$Q_{17} =$
{(2 5 0 5), (4 0 0 4), (1 6 0 6), (2 0 0 2), (4 3 0 3), (3 0 0 3), (6 1 0 1), (5 0 0 5), (3 4 0 4), (6 0 0 6), (5 2 0

$Q_{18} =$
{(1 0 0 6), (3 6 0 5), (2 6 0 4), (6 0 0 6), (4 1 0 2), (5 1 0 3), (3 4 0 2), (2 3 0 3), (6 6 0 1), (4 3 0 5), (5 4 0

$Q_{19} =$
{(5 6 0 3), (4 6 0 2), (6 0 0 6), (2 1 0 4), (3 1 0 5), (3 3 0 2), (1 6 0 6), (5 3 0 4), (4 4 0 5), (6 1 0 1), (2 4 0

$Q_{20} =$
{(3 4 0 5), (2 4 0 4), (6 0 0 6), (4 3 0 2), (5 3 0 3), (3 5 0 2), (6 4 0 1), (5 5 0 4), (4 2 0 5), (1 3 0 6), (2 2 0

$Q_{21} =$
{(2 0 0 3), (4 0 0 2), (1 0 0 6), (2 0 0 4), (4 0 0 5), (6 0 0 1), (5 0 0 3), (3 0 0 2), (6 0 0 6), (5 0 0 4), (3 0 0

$Q_{22} =$
{(2 3 0 5), (4 0 0 4), (1 5 0 6), (2 0 0 2), (4 6 0 3), (6 2 0 1), (5 0 0 5), (3 1 0 4), (6 0 0 6), (5 4 0 2), (3 0 0

$Q_{23} =$
{(2 0 0 5), (4 0 0 4), (1 0 0 6), (2 0 0 2), (4 0 0 3), (3 0 0 3), (6 0 0 1), (5 0 0 5), (3 0 0 4), (6 0 0 6), (5 0 0

$Q_{24} =$
{(1 3 0 5), (1 4 0 4), (1 2 0 6), (1 6 0 2), (1 5 0 3), (6 3 0 3), (6 5 0 1), (6 1 0 5), (6 2 0 4), (6 0 0 6), (6 4 0

$Q_{25} =$
{(3 2 0 2), (2 3 0 4), (6 3 0 1), (4 4 0 2), (5 2 0 4), (4 5 0 5), (5 4 0 3), (1 4 0 6), (3 3 0 5), (2 5 0 3), (6 0 0

$Q_{26} =$
{(4 2 0 6), (2 6 0 1), (1 5 0 6), (4 4 0 1), (2 4 0 6), (5 1 0 6), (6 2 0 1), (3 3 0 6), (5 3 0 1), (6 0 0 6), (3 5 0

$Q_{27} =$
{(5 5 0 2), (4 0 0 4), (6 6 0 1), (2 0 0 2), (3 3 0 4), (5 0 0 5), (4 4 0 3), (6 0 0 6), (2 2 0 5), (3 0 0 3), (1 1 0

$Q_{28} =$
{(2 4 0 5), (4 0 0 4), (1 2 0 6), (2 0 0 2), (4 1 0 3), (3 0 0 3), (6 5 0 1), (5 0 0 5), (3 6 0 4), (6 0 0 6), (5 3 0

$Q_1 \supseteq M_1, M_{46}, M_{54}$; $Q_2 \supseteq M_2, M_{45}, M_{53}$; $Q_3 \supseteq M_3, M_{58}, M_{68}$
$Q_4 \supseteq M_4, M_{57}, M_{71}$; $Q_5 \supseteq M_5, M_{17}, M_{24}$; $Q_6 \supseteq M_6, M_{51}, M_{59}$
$Q_7 \supseteq M_7, M_{52}, M_{60}$; $Q_8 \supseteq M_8, M_{43}, M_{61}$; $Q_9 \supseteq M_9, M_{42}, M_{63}$
$Q_{10} \supseteq M_{10}, M_{27}, M_{44}$; $Q_{11} \supseteq M_{11}, M_{28}, M_{47}$; $Q_{12} \supseteq M_{12}, M_{31}, M_{62}$
$Q_{13} \supseteq M_{13}, M_{29}, M_{49}$; $Q_{14} \supseteq M_{14}, M_{30}, M_{48}$; $Q_{15} \supseteq M_{15}, M_{26}, M_{65}$
$Q_{16} \supseteq M_5, M_{16}, M_{23}$; $Q_{17} \supseteq M_5, M_{36}, M_{41}$; $Q_{18} \supseteq M_{18}, M_{21}, M_{75}$
$Q_{19} \supseteq M_{19}, M_{22}, M_{70}$; $Q_{20} \supseteq M_{20}, M_{32}, M_{66}$; $Q_{21} \supseteq M_{33}, M_{34}, M_{50}$
$Q_{22} \supseteq M_5, M_{64}, M_{74}$; $Q_{23} \supseteq M_5, M_{39}, M_{40}$; $Q_{24} \supseteq M_{55}, M_{72}, M_{77}$
$Q_{25} \supseteq M_{25}, M_{35}, M_{56}$; $Q_{26} \supseteq M_{69}, M_{73}, M_{78}$; $Q_{27} \supseteq M_5, M_{37}, M_{38}$
$Q_{28} \supseteq M_5, M_{67}, M_{76}$.

And

$L_1 \subseteq Q_1, Q_2, Q_{21}, Q_{23}$; $L_2 \subseteq Q_4, Q_{11}, Q_{18}, Q_{27}$; $L_3 \subseteq Q_3, Q_{10}, Q_{17}, Q_{19}$
$L_4 \subseteq Q_{14}, Q_{15}, Q_{24}, Q_{28}$; $L_5 \subseteq Q_{12}, Q_{13}, Q_{22}, Q_{26}$; $L_6 \subseteq Q_6, Q_8, Q_{16}, Q_{20}$
$L_7 \subseteq Q_5, Q_7, Q_9, Q_{25}$

The fourteen -element subgroups of G -

R_1 to R_3 are the subgroups of order 14, where,

$R_1 =$
{(1 1 0 1), (1 2 0 1), (1 3 0 1), (1 4 0 1), (1 5 0 1), (1 6 0 1), (6 0 0 6), (6 1 0 6), (6 2 0 6), (6 3 0 6), (6 4 0

R_2
= {(1 1 0 1), (1 2 0 1), (1 3 0 1), (1 4 0 1), (1 5 0 1), (1 6 0 1), (1 1 0 6), (1 2 0 6), (1 3 0 6), (1 4 0 6), (1 5

R_3
= {(1 1 0 1), (1 2 0 1), (1 3 0 1), (1 4 0 1), (1 5 0 1), (1 6 0 1), (6 6 0 1), (6 5 0 1), (6 4 0 1), (6 3 0 1), (6 2

And

$R_1 \supseteq N_1, H_7$; $R_2 \supseteq N_1, H_1, H_3, H_6, H_8, H_{10}, H_{12}, H_{14}$, $R_3 \supseteq N_1, H_2, H_4, H_5, H_9, H_{11}, H_{13}, H_{15}$

The eighteen -element subgroups of G -

S_1 to S_{21} are the subgroups of order 14, where,

S_1
$= \{(1\,0\,0\,2), (1\,0\,0\,3), (1\,0\,0\,4), (1\,0\,0\,5), (2\,0\,0\,1), (2\,0\,0\,2), (2\,0\,0\,3), (2\,0\,0\,4), (2\,0\,0\,5), (2\,0\,0\,6), (4\,0$

S_2
$= \{(3\,0\,0\,1), (2\,0\,0\,1), (6\,0\,0\,1), (4\,0\,0\,1), (5\,0\,0\,1), (2\,0\,0\,2), (6\,0\,0\,2), (4\,0\,0\,2), (5\,0\,0\,2), (1\,0\,0\,2), (6\,0$

S_3
$= \{(1\,1\,0\,3), (1\,4\,0\,2), (1\,6\,0\,6), (1\,5\,0\,4), (1\,2\,0\,5), (2\,5\,0\,5), (2\,3\,0\,1), (2\,4\,0\,3), (2\,0\,0\,2), (2\,2\,0\,6), (2\,1$

S_4
$= \{(3\,1\,0\,1), (2\,4\,0\,1), (6\,6\,0\,1), (4\,5\,0\,1), (5\,2\,0\,1), (3\,3\,0\,4), (2\,6\,0\,4), (6\,1\,0\,4), (4\,0\,0\,4), (5\,4\,0\,4), (3\,4$

S_5
$= \{(3\,0\,0\,3), (2\,0\,0\,2), (6\,0\,0\,6), (4\,0\,0\,4), (5\,0\,0\,5), (4\,0\,0\,2), (5\,0\,0\,3), (1\,0\,0\,4), (3\,0\,0\,5), (2\,0\,0\,1), (3\,0$

S_6
$= \{(3\,3\,0\,1), (2\,5\,0\,1), (6\,4\,0\,1), (4\,1\,0\,1), (5\,6\,0\,1), (2\,4\,0\,4), (6\,3\,0\,4), (4\,0\,0\,4), (5\,5\,0\,4), (1\,6\,0\,4), (6\,6$

S_7
$= \{(1\,3\,0\,3), (1\,5\,0\,2), (1\,4\,0\,6), (1\,1\,0\,4), (1\,6\,0\,5), (2\,3\,0\,4), (2\,1\,0\,5), (2\,2\,0\,1), (2\,5\,0\,3), (2\,0\,0\,2), (4\,2$

S_8
$= \{(1\,1\,0\,5), (1\,6\,0\,4), (1\,3\,0\,6), (1\,4\,0\,3), (4\,2\,0\,5), (2\,5\,0\,1), (2\,6\,0\,5), (2\,4\,0\,4), (2\,1\,0\,6), (2\,0\,0\,2), (4\,4$

$S_9 =$
$\{(5\,1\,0\,1), (4\,6\,0\,1), (6\,3\,0\,1), (2\,2\,0\,1), (3\,4\,0\,1), (3\,2\,0\,2), (1\,5\,0\,2), (5\,6\,0\,2), (6\,1\,0\,2), (1\,1\,0\,4), (5\,2\,0$

$S_{10} =$
$\{(5\,5\,0\,1), (4\,2\,0\,1), (6\,1\,0\,1), (2\,3\,0\,1), (3\,6\,0\,1), (5\,3\,0\,4), (4\,0\,0\,4), (6\,6\,0\,4), (2\,1\,0\,4), (3\,4\,0\,4), (1\,5\,0$

S_{11}
$= \{(1\,5\,0\,5), (1\,2\,0\,4), (1\,1\,0\,6), (1\,3\,0\,2), (1\,6\,0\,3), (4\,3\,0\,5), (4\,0\,0\,4), (4\,6\,0\,6), (4\,1\,0\,2), (4\,4\,0\,3), (4$

S_{12}
$= \{(1\,2\,0\,3), (1\,1\,0\,2), (1\,5\,0\,6), (1\,3\,0\,4), (1\,4\,0\,5), (2\,1\,0\,3), (2\,0\,0\,2), (2\,4\,0\,6), (2\,2\,0\,4), (2\,3\,0\,5), (2$

S_{13}
$= \{(3\,1\,0\,2), (2\,5\,0\,4), (6\,5\,0\,1), (4\,2\,0\,2), (5\,1\,0\,4), (2\,1\,0\,1), (6\,4\,0\,2), (4\,0\,0\,4), (5\,4\,0\,1), (1\,6\,0\,2), (3$

S_{14}
$= \{(5\,4\,0\,3), (4\,4\,0\,2), (6\,0\,0\,6), (2\,3\,0\,4), (3\,3\,0\,5), (1\,1\,0\,4), (5\,0\,0\,5), (4\,6\,0\,1), (6\,6\,0\,3), (2\,0\,0\,2), (3\,1$

S_{15}
$= \{(4\,5\,0\,6), (2\,1\,0\,1), (1\,2\,0\,6), (4\,3\,0\,1), (2\,3\,0\,6), (4\,2\,0\,2), (2\,4\,0\,5), (1\,6\,0\,2), (4\,6\,0\,5), (2\,0\,0\,2), (1\,3$

S_{16}
$= \{(5\,6\,0\,4), (4\,5\,0\,2), (6\,2\,0\,1), (2\,2\,0\,4), (3\,6\,0\,2), (1\,1\,0\,2), (5\,3\,0\,1), (4\,0\,0\,4), (6\,3\,0\,2), (2\,6\,0\,1), (3\,1$

S_{17}
$= \{(5\,4\,0\,6), (4\,2\,0\,1), (6\,0\,0\,6), (2\,3\,0\,1), (3\,5\,0\,6), (1\,4\,0\,2), (5\,0\,0\,5), (4\,6\,0\,2), (6\,3\,0\,5), (2\,0\,0\,2), (3\,1$

S_{18}
$= \{(3\,4\,0\,5), (2\,4\,0\,4), (6\,0\,0\,6), (4\,3\,0\,2), (5\,3\,0\,3), (2\,5\,0\,1), (6\,5\,0\,5), (4\,0\,0\,4), (5\,2\,0\,6), (1\,2\,0\,2), (3\,0$

S_{19}
$= \{(3\,6\,0\,5), (2\,6\,0\,4), (6\,0\,0\,6), (4\,1\,0\,2), (5\,1\,0\,3), (2\,4\,0\,1), (6\,4\,0\,5), (4\,0\,0\,4), (5\,3\,0\,6), (1\,3\,0\,2), (3\,0$

S_{20}
$= \{(5\,1\,0\,6), (4\,4\,0\,1), (6\,0\,0\,6), (2\,6\,0\,1), (3\,3\,0\,6), (1\,1\,0\,2), (5\,0\,0\,5), (4\,5\,0\,2), (6\,6\,0\,5), (2\,0\,0\,2), (3\,2$

S_{21}
$= \{(3\ 5\ 0\ 5\ ), (2\ 5\ 0\ 4\ ), (6\ 0\ 0\ 6\ ), (4\ 2\ 0\ 2\ ), (5\ 2\ 0\ 3\ ), (2\ 1\ 0\ 1\ ), (6\ 1\ 0\ 5\ ), (4\ 0\ 0\ 4\ ), (5\ 6\ 0\ 6\ ), (1\ 6\ 0\ 2\ ), (3$
And

$S_1 \supseteq M_1, M_{33}, M_{39}, M_{45}, P_1$; $S_2 \supseteq M_2, M_{34}, M_{40}, M_{46}, P_1$
$S_3 \supseteq M_3, M_{19}, M_{36}, M_{44}, P_6$; $S_4 \supseteq M_4, M_{18}, M_{37}, M_{47}, P_7$
$S_5 \supseteq M_5, M_{50}, M_{53}, M_{54}, P_1$; $S_6 \supseteq M_6, M_{16}, M_{43}, M_{66}, P_4$
$S_7 \supseteq M_7, M_{17}, M_{42}, M_{56}, P_3$; $S_8 \supseteq M_8, M_{23}, M_{32}, M_{59}, P_4$
$S_9 \supseteq M_9, M_{24}, M_{35}, M_{60}, P_3$; $S_{10} \supseteq M_{10}, M_{22}, M_{41}, M_{58}, P_6$
$S_{11} \supseteq M_{11}, M_{21}, M_{38}, M_{57}, P_7$; $S_{12} \supseteq M_{12}, M_{13}, M_{64}, M_{73}, P_2$
$S_{13} \supseteq M_{14}, M_{15}, M_{67}, M_{72}, P_5$; $S_{14} \supseteq M_5, M_{25}, M_{52}, M_{63}, P_3$
$S_{15} \supseteq M_{26}, M_{30}, M_{76}, M_{77}, P_5$; $S_{16} \supseteq M_{29}, M_{31}, M_{74}, M_{78}, P_2$
$S_{17} \supseteq M_5, M_{27}, M_{68}, M_{70}, P_6$; $S_{18} \supseteq M_5, M_{20}, M_{51}, M_{61}, P_4$
$S_{19} \supseteq M_5, M_{28}, M_{71}, M_{75}, P_7$; $S_{20} \supseteq M_5, M_{49}, M_{62}, M_{69}, P_2$
$S_{21} \supseteq M_5, M_{48}, M_{55}, M_{65}, P_5$;

<u>The twenty one -element subgroups of G -</u>

T_1 to T_4 are the subgroups of order 21, where,
T_1
$= \{(1\ 1\ 0\ 1\ ), (1\ 2\ 0\ 1\ ), (1\ 3\ 0\ 1\ ), (1\ 4\ 0\ 1\ ), (1\ 5\ 0\ 1\ ), (1\ 6\ 0\ 1\ ), (2\ 1\ 0\ 2\ ), (4\ 6\ 0\ 4\ ), (2\ 0\ 0\ 2\ ), (4\ 4\ 0\ 4\ ), (2$
T_2
$= \{(1\ 1\ 0\ 1\ ), (1\ 2\ 0\ 1\ ), (1\ 3\ 0\ 1\ ), (1\ 4\ 0\ 1\ ), (1\ 5\ 0\ 1\ ), (1\ 6\ 0\ 1\ ), (1\ 1\ 0\ 2\ ), (1\ 2\ 0\ 2\ ), (1\ 3\ 0\ 2\ ), (1\ 4\ 0\ 2\ ), (1$
T_3
$= \{(1\ 1\ 0\ 1\ ), (1\ 2\ 0\ 1\ ), (1\ 3\ 0\ 1\ ), (1\ 4\ 0\ 1\ ), (1\ 5\ 0\ 1\ ), (1\ 6\ 0\ 1\ ), (2\ 0\ 0\ 1\ ), (2\ 1\ 0\ 4\ ), (2\ 0\ 0\ 2\ ), (4\ 4\ 0\ 4\ ), (2$
T_4
$= \{(1\ 1\ 0\ 1\ ), (1\ 2\ 0\ 1\ ), (1\ 3\ 0\ 1\ ), (1\ 4\ 0\ 1\ ), (1\ 5\ 0\ 1\ ), (1\ 6\ 0\ 1\ ), (2\ 1\ 0\ 4\ ), (2\ 2\ 0\ 4\ ), (2\ 3\ 0\ 4\ ), (2\ 4\ 0\ 4\ ), (2$
And

$T_1 \supseteq N_1, K_5$; $T_2 \supseteq N_1, K_1, K_3, K_7, K_8, K_{11}, K_{13}, K_{22}$,
$T_3 \supseteq N_1, K_2, K_4, K_6, K_9, K_{10}, K_{12}, K_{21}$; $T_4 \supseteq N_1, K_{14}, K_{15}, K_{16}, K_{17}, K_{18}, K_{19}, K_{20}$

<u>The twenty eight -element subgroup of G -</u>

U_1 is the subgroup of order 28, where,
U_1
$= \{(1\ 1\ 0\ 1\ ), (1\ 2\ 0\ 1\ ), (1\ 3\ 0\ 1\ ), (1\ 4\ 0\ 1\ ), (1\ 5\ 0\ 1\ ), (1\ 6\ 0\ 1\ ), (1\ 0\ 0\ 6\ ), (6\ 0\ 0\ 1\ ), (1\ 1\ 0\ 6\ ),, (1\ 2\ 0\ 6\ ), (1$
And
$U_1 \supseteq N_1, R_1, R_2, R_3, L_1, L_2, L_3, L_4, L_5, L_6, L_7$.

<u>The thirty sixth -element of subgroups of G -</u>

V_1 to V_7 are the subgroups of order 36, where,
V_1
$= \{(1\ 0\ 0\ 2\ ), (1\ 0\ 0\ 3\ ), (1\ 0\ 0\ 4\ ), (1\ 0\ 0\ 5\ ), (1\ 0\ 0\ 6\ ), (2\ 0\ 0\ 1\ ), (2\ 0\ 0\ 2\ ), (2\ 0\ 0\ 3\ ), (2\ 0\ 0\ 4\ ),, (2\ 0\ 0\ 5\ ), (2$
V_2
$= \{(1\ 1\ 0\ 3\ ), (1\ 4\ 0\ 2\ ), (1\ 6\ 0\ 6\ ), (1\ 5\ 0\ 4\ ), (1\ 2\ 0\ 5\ ), (2\ 5\ 0\ 5\ ), (2\ 3\ 0\ 1\ ), (2\ 4\ 0\ 3\ ), (2\ 0\ 0\ 2\ ),, (2\ 2\ 0\ 6\ ), (2$
V_3
$= \{(3\ 1\ 0\ 1\ ), (2\ 4\ 0\ 1\ ), (6\ 6\ 0\ 1\ ), (4\ 5\ 0\ 1\ ), (5\ 2\ 0\ 1\ ), (3\ 3\ 0\ 4\ ), (2\ 6\ 0\ 4\ ), (6\ 1\ 0\ 4\ ),, (4\ 0\ 0\ 4\ ), (5\ 4\ 0\ 4\ ), (3$

V_4
$= \{(3\,3\,0\,1), (2\,5\,0\,1), (6\,4\,0\,1), (4\,1\,0\,1), (5\,6\,0\,1), (2\,4\,0\,4), (6\,3\,0\,4), (4\,0\,0\,4), , (5\,5\,0\,4), (1\,6\,0\,4), (6$
V_5
$= \{(1\,3\,0\,3), (1\,5\,0\,2), (1\,4\,0\,6), (1\,1\,0\,4), (1\,6\,0\,5), (2\,3\,0\,4), (2\,1\,0\,5), (2\,2\,0\,1), , (2\,5\,0\,3), (2\,0\,0\,2), (4$
V_6
$= \{(1\,2\,0\,3), (1\,1\,0\,2), (1\,5\,0\,6), (1\,3\,0\,4), (1\,4\,0\,5), (2\,1\,0\,3), (2\,0\,0\,2), (2\,4\,0\,6), , (2\,2\,0\,4), (2\,3\,0\,5), (2$
V_7
$= \{(3\,1\,0\,2), (2\,5\,0\,4), (6\,5\,0\,1), (4\,2\,0\,2), (5\,1\,0\,4), (2\,1\,0\,1), (6\,4\,0\,2), (4\,0\,0\,4), , (5\,4\,0\,1), (1\,6\,0\,2), (3$
And
$V_1 \supseteq S_1, S_2, S_5$ and Q_1, Q_2, Q_{21}, Q_{23}; $V_2 \supseteq S_3, S_{10}, S_{17}$ and $Q_3, Q_{10}, Q_{17}, Q_{19}$;
$V_3 \supseteq S_4, S_{11}, S_{19}$ and $Q_4, Q_{11}, Q_{18}, Q_{27}$; $V_4 \supseteq S_6, S_8, S_{18}$ and Q_6, Q_8, Q_{16}, Q_{20};
$V_5 \supseteq S_7, S_9, S_{14}$ and Q_5, Q_7, Q_9, Q_{25}; $V_6 \supseteq S_{12}, S_{16}, S_{20}$ and $Q_{12}, Q_{13}, Q_{22}, Q_{26}$;
$V_7 \supseteq S_{13}, S_{15}, S_{21}$ and $Q_{14}, Q_{15}, Q_{24}, Q_{28}$;

The forty second -element of subgroups of G -

W_1 to W_{12} are the subgroups of order 42, where,
W_1
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (1\,0\,0\,2), (1\,0\,0\,3), (1\,0\,0\,4), (1\,0\,0\,5), (1\,($
W_2
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (3\,0\,0\,1), (2\,0\,0\,1), (6\,0\,0\,1), (4\,0\,0\,1), (5\,($
W_3
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (3\,0\,0\,3), (2\,0\,0\,2), (4\,0\,0\,4), (5\,0\,0\,5), (6\,($
W_4
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (2\,1\,0\,3), (4\,5\,0\,2), (1\,5\,0\,6), (2\,2\,0\,4), (4\,1$
W_5
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (3\,1\,0\,2), (2\,5\,0\,4), (6\,5\,0\,1), (4\,2\,0\,2), (5\,1$
W_6
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (3\,2\,0\,4), (2\,0\,0\,2), (6\,4\,0\,1), (4\,0\,0\,4), (5\,1$
W_7
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (4\,2\,0\,3), (2\,0\,0\,2), (1\,4\,0\,6), (4\,0\,0\,4), (2\,1$
W_8
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (3\,4\,0\,5), (2\,4\,0\,4), (6\,0\,0\,6), (4\,3\,0\,2), (5\,3$
W_9
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (4\,5\,0\,6), (2\,1\,0\,1), (1\,2\,0\,6), (4\,3\,0\,1), (2\,3$
W_{10}
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (5\,4\,0\,6), (4\,2\,0\,1), (6\,0\,0\,6), (2\,3\,0\,1), (3\,5$
W_{11}
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (6\,4\,0\,5), (1\,2\,0\,4), (6\,0\,0\,6), (1\,3\,0\,2), (6\,5$
W_{12}
$= \{(1\,1\,0\,1), (1\,2\,0\,1), (1\,3\,0\,1), (1\,4\,0\,1), (1\,5\,0\,1), (1\,6\,0\,1), (6\,5\,0\,4), (1\,1\,0\,2), (6\,2\,0\,1), (1\,3\,0\,4), (6\,3$
And
$W_1 \supseteq M_1, M_3, M_7, M_8, M_{11}, M_{12}, M_{77}, R_2, T_2$;
$W_2 \supseteq M_2, M_4, M_6, M_9, M_{10}, M_{15}, M_{78}, R_3, T_3$
$W_3 \supseteq M_5, R_1, T_1$
$W_4 \supseteq M_{13}, M_{19}, M_{21}, M_{30}, M_{32}, M_{33}, M_{56}, R_2, T_4$

International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 14S, 2025
https://www.theaspd.com/ijes.php

$W_5 \supseteq M_{14}, M_{18}, M_{22}, M_{29}, M_{34}, M_{35}, M_{66}, R_3, T_4$;
$W_6 \supseteq M_{16}, M_{24}, M_{37}, M_{40}, M_{41}, M_{67}, M_{74}, R_3, T_1$
$W_7 \supseteq M_{17}, M_{23}, M_{36}, M_{38}, M_{39}, M_{64}, M_{76}, R_2, T_1$
$W_8 \supseteq M_{20}, M_{25}, M_{48}, M_{49}, M_{50}, M_{70}, M_{75}, R_1, T_2$;
$W_9 \supseteq M_{26}, M_{42}, M_{44}, M_{45}, M_{57}, M_{59}, M_{73}, R_2, T_3$
$W_{10} \supseteq M_{27}, M_{51}, M_{53}, M_{63}, M_{65}, M_{69}, M_{51}, R_1, T_3$
$W_{11} \supseteq M_{28}, M_{52}, M_{54}, M_{55}, M_{61}, M_{62}, M_{68}, R_1, T_2$;
$W_{12} \supseteq M_{31}, M_{43}, M_{46}, M_{47}, M_{58}, M_{60}, M_{72}, R_3, T_2$;

The sixty third -element subgroup of G -

X_1 is the subgroups of order 63, where,

$X_1 = \{(4\ 4\ 0\ 2), (1\ 2\ 0\ 4), (2\ 0\ 0\ 1), (2\ 6\ 0\ 1), (2\ 4\ 0\ 1), (2\ 1\ 0\ 1), (2\ 3\ 0\ 1), (2\ 5\ 0\ 1), (2\ 2\ 0\ 1), (4\ 2\ 0\ 2), (4\ 5$

And

$X_1 \supseteq T_1, T_2, T_3, T_4$

The eighty four -element subgroups of G -

Y_1 to Y_4 are the subgroups of order 84, where,

$Y_1 = \{(1\ 1\ 0\ 1), (1\ 2\ 0\ 1), (1\ 3\ 0\ 1), (1\ 4\ 0\ 1), (1\ 5\ 0\ 1), (1\ 6\ 0\ 1), (1\ 0\ 0\ 6), (6\ 0\ 0\ 1), (1\ 1\ 0\ 6), (1\ 2\ 0\ 6), (1\ 3$
.

$Y_2 = \{(5\ 1\ 0\ 1), (3\ 2\ 0\ 1),, (2\ 2\ 0\ 1), (6\ 2\ 0\ 1), (4\ 1\ 0\ 1), (5\ 2\ 0\ 1), (3\ 3\ 0\ 1), (2\ 3\ 0\ 1), (6\ 3\ 0\ 1), (4\ 3\ 0\ 1)\ (5\ 3\ 0\ 1), ($

$Y_3 = \{(1\ 1\ 0\ 1), (1\ 2\ 0\ 1), (1\ 3\ 0\ 1), (1\ 4\ 0\ 1), (1\ 5\ 0\ 1), (1\ 6\ 0\ 1), (3\ 0\ 0\ 3), (2\ 0\ 0\ 2), (4\ 0\ 0\ 4), (5\ 0\ 0\ 5), (6\ 0$

$Y_4 = \{(1\ 1\ 0\ 1), (1\ 2\ 0\ 1), (1\ 3\ 0\ 1), (1\ 4\ 0\ 1), (1\ 5\ 0\ 1), (1\ 6\ 0\ 1), (2\ 1\ 0\ 3), (4\ 5\ 0\ 2), (1\ 5\ 0\ 6), (2\ 2\ 0\ 4), (4\ 1$

And

$Y_1 \supseteq W_1, W_{11}, W_{12}$; $Y_2 \supseteq W_2, W_9, W_{10}$; $Y_3 \supseteq W_3, W_6, W_7$ and $Y_4 \supseteq W_4, W_5, W_8$.

The one twenty six -element subgroups of G -

Z_1 to Z_3 are the subgroups of order 84, where,

$Z_1 = \{(1\ 1\ 0\ 1), (1\ 2\ 0\ 1), (1\ 3\ 0\ 1), (1\ 4\ 0\ 1), (1\ 5\ 0\ 1), (1\ 6\ 0\ 1), (1\ 1\ 0\ 2), (1\ 2\ 0\ 2), (1\ 3\ 0\ 2), (1\ 4\ 0\ 2), (1\ 5$

$Z_2 = \{(1\ 1\ 0\ 1), (1\ 2\ 0\ 1), (1\ 3\ 0\ 1), (1\ 4\ 0\ 1), (1\ 5\ 0\ 1), (1\ 6\ 0\ 1), (1\ 1\ 0\ 2), (1\ 2\ 0\ 2), (1\ 3\ 0\ 2), (1\ 4\ 0\ 2), (1\ 5$

$Z_3 = \{(1\ 1\ 0\ 1), (1\ 2\ 0\ 1), (1\ 3\ 0\ 1), (1\ 4\ 0\ 1), (1\ 5\ 0\ 1), (1\ 6\ 0\ 1), (1\ 1\ 0\ 2), (1\ 2\ 0\ 2), (1\ 3\ 0\ 2), (1\ 4\ 0\ 2), (1\ 5$

And

$Z_1 \supseteq W_1, W_4, W_7, W_9$; $Z_2 \supseteq W_2, W_5, W_6, W_{12}$; $Z_3 \supseteq W_3, W_8, W_{10}, W_{11}$.

We tabulate the subgroups of G, when p=7 in the order in which they lie in different maximal subgroups(co-atoms).

Intervals [{e}, V_i] in $L_u(G)$, i= 1 to 7

Order	Subgroups

36	V_1
18	S_1, S_2, S_5
12	Q_1, Q_2, Q_{21}, Q_{23}
9	P_1
6	M_1, M_2, M_5, M_{33}, M_{34}, M_{39}, M_{40}, M_{45}, M_{46}, M_{50}, M_{53}, M_{54}
4	L_1
3	K_1, K_2, K_5, K_{15}
2	H_1, H_2, H_7
1	{e}

Order	Subgroups
36	V_2
18	S_3, S_{10}, S_{17}
12	Q_3, Q_{10}, Q_{17}, Q_{19}
9	P_6
6	M_3, M_5, M_{10}, M_{19}, M_{22}, M_{27}, M_{36}, M_{41}, M_{44}, M_{58}, M_{68}, M_{70}
4	L_3
3	K_5, K_{12}, K_{19}, K_{22}
2	H_4, H_6, H_7
1	{e}

Order	Subgroups
36	V_3
18	S_4, S_{11}, S_{19}
12	Q_4, Q_{11}, Q_{18}, Q_{27}
9	P_7
6	M_4, M_5, M_{11}, M_{18}, M_{21}, M_{28}, M_{37}, M_{38}, M_{47}, M_{57}, M_{71}, M_{75}
4	L_2
3	K_5, K_{13}, K_{20}, K_{21}
2	H_3, H_5, H_7
1	{e}

Order	Subgroups
36	V_4
18	S_6, S_8, S_{18}
12	Q_6, Q_8, Q_{16}, Q_{20}
9	P_4

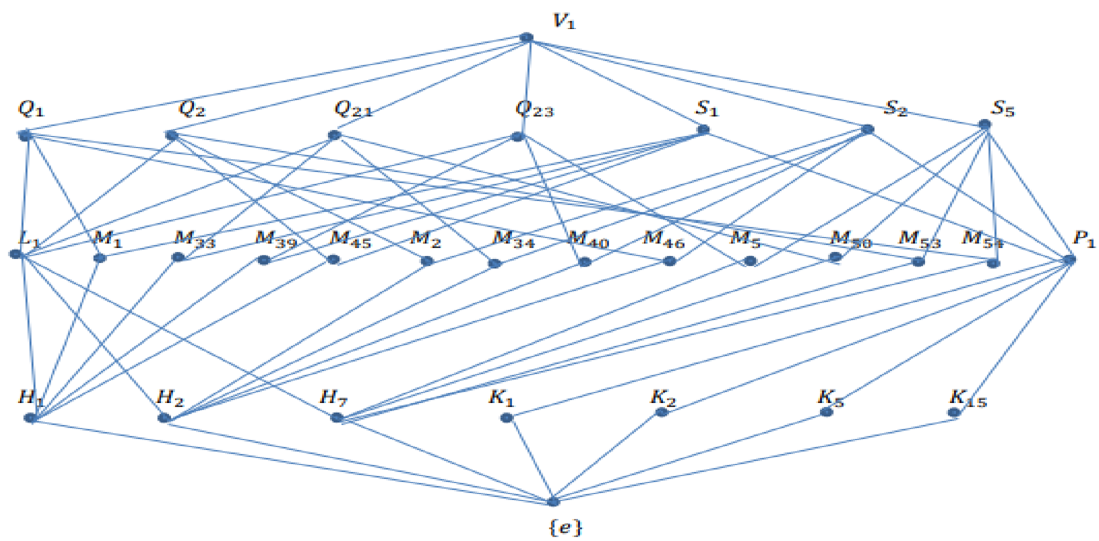
International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 14S, 2025
https://www.theaspd.com/ijes.php

6	M_5, M_6, M_8, M_{16}, M_{20}, M_{23}, M_{32}, M_{43}, M_{51}, M_{59}, M_{61}, M_{66}
4	L_6
3	K_5, K_7, K_9, K_{16}
2	H_3, H_7, H_{10}
1	{e}
Order	Subgroups
36	V_5
18	S_7, S_9, S_{14}
12	Q_5, Q_7, Q_9, Q_{25}
9	P_3
6	M_5, M_{17}, M_{24}, M_7, M_{52}, M_{60}, M_9, M_{42}, M_{63}, M_{25}, M_{35}, M_{56}
4	L_7
3	K_5, K_6, K_8, K_{14}
2	H_7, H_{11}, H_{12}
1	{e}
Order	Subgroups
36	V_6
18	S_{12}, S_{16}, S_{20}
12	Q_{12}, Q_{13}, Q_{22}, Q_{26}
9	P_2
6	M_{12}, M_{31}, M_{62}, $M1_3$, M_{29}, M_{49}, M_5, M_{64}, M_{74}, M_{69}, M_{73}, M_{78}
4	L_5
3	K_3, K_5, K_{16}, K_{17}
2	H_7, H_9, H_{14}
1	{e}
Order	Subgroups
36	V_7
18	S_{13}, S_{15}, S_{21}
12	Q_{14}, Q_{15}, Q_{24}, Q_{28}
9	P_5
6	M_{14}, M_{30}, M_{48}, M_{15}, M_{26}, M_{65}, M_{55}, M_{72}, M_{77}, M_5, M_{67}, M_{76}
4	L_4
3	K_4, K_5, K_{11}, K_{18}
2	H_7, H_8, H_{15}

1	{e}

Each V_i is of order 36.

We observe that the number of subgroups of orders 2,3,4,6,9,12,18 and 36 is 3,4,1,12,1,4,3 and 1 respectively and the lattice structure of the intervals[{e},V_i] are isomorphic. Typically, we display it for V_i as given below.



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