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Normal Spaces In The Context Of Pythagorean Fuzzy Nano $\delta\beta$-Open Sets

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In this paper, we introduce a new class of spaces referred to as Pythagorean fuzzy nano normal spaces, including their variants: δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$-normal spaces. We also define their corresponding strongly Pythagorean fuzzy nano normal spaces. These spaces are formulated using the respective types of Pythagorean fuzzy nano open sets within the framework of Pythagorean fuzzy nano topological spaces. Furthermore, we explore the relationships among these newly defined spaces and their connections with existing spaces. Additionally, we study their basic properties and provide characterizations of the introduced normal spaces.

Keywords: *Pythagorean fuzzy nano $\delta\beta$-open, Pythagorean fuzzy nano $\delta\beta$-closed, Pythagorean fuzzy nano $\delta\beta$ normal space, strongly Pythagorean fuzzy nano $\delta\beta$ normal space.*
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INTRODUCTION

The concept of fuzzy sets, first introduced by Zadeh in 1965 [41], has found wide-ranging applications in fields such as decision theory, artificial intelligence, operations research, expert systems, computer science, data analytics, pattern recognition, management science, and robotics. In 1968, Chang and Warren [15, 35] extended this concept by introducing fuzzy topological spaces (FTS), incorporating fundamental topological notions such as open and closed sets, neighborhoods, interiors, closures, continuity, and compactness. Subsequent studies further explored the applications of fuzzy sets across various domains [1, 13, 27, 32]. Over time, numerous specialized fuzzy topological structures have been developed to address specific theoretical and practical needs.

In 1997, Dogan Coker [9, 18, 22] introduced the concept of intuitionistic fuzzy topological spaces and explored their properties related to compactness and continuity. Building upon intuitionistic fuzzy setsâ€”which account for both membership and non-membership degreesâ€”Pythagorean fuzzy sets (PFS) have gained attention due to their broader applicability [32, 29]. While both set types incorporate membership (μ) and non-membership (λ) degrees, their constraints differ: intuitionistic fuzzy sets satisfy $\mu + \lambda \leq 1$, whereas Pythagorean fuzzy sets satisfy $\mu^2 + \lambda^2 \leq 1$.

To offer greater flexibility in uncertainty modeling, Yager [38] introduced non-standard fuzzy sets in 2013, comparing them with intuitionistic fuzzy sets and proposing the Pythagorean fuzzy set (PFS) as an effective model in decision-making scenarios [3, 40, 39]. PFS has since been applied in areas such as job placement based on academic performance [23] and mask selection during the COVID-19 pandemic using the Pythagorean TOPSIS technique [26].

Subsequently, Murat et al. [21] extended the fuzzy topological framework by developing Pythagorean fuzzy topological spaces (PFTS), inspired by classical fuzzy topological spaces (FTS) [19, 20, 25], and defined Pythagorean fuzzy continuous functions between such spaces.

In parallel developments, Saha [28] introduced the concept of δ-open sets in fuzzy topological spaces. This concept was further extended in 2019 by Acikgoz and Esenbel [2], who introduced neutrosophic soft δ-topologies. Further contributions were made by Aranganayagi et al., Surendra et al., and Vadivel et al. [7, 8, 30, 31, 33, 34], who investigated δ-open sets in neutrosophic, neutrosophic soft, neutrosophic hypersoft, and neutrosophic nano topological spaces, particularly focusing on their mappings and separation axioms.

Similarity measures have emerged as essential tools for quantifying vagueness and evaluating the closeness between fuzzy sets. In this context, Zhang [16] proposed similarity-based techniques for Pythagorean fuzzy multi-attribute decision-making. Peng et al. [24] introduced several new distance and similarity measures aimed at solving problems in pattern recognition, medical diagnosis, and clustering analysis, also examining their transformation properties. Wei and Wei [36] further developed cosine-based similarity functions for decision-making applications.

However, several existing PFS similarity and distance measures suffer from limitations such as division-by-zero issues, inability to distinguish between positive and negative differences, and non-compliance with core axioms (e.g., the third and fourth similarity axioms). These counter-intuitive behaviors [36, 16, 24] hinder the decision-maker's (DMâ€™s) ability to identify optimal or convincing alternatives.

The objective of this paper is to address these challenges by proposing a novel similarity measure for Pythagorean fuzzy sets that overcomes these counter-intuitive limitations and provides a more robust decision-making tool.

Research Gap: To date, no studies have been reported in the literature on Pythagorean fuzzy nano topological spaces that investigate newly defined spaces such as Pythagorean fuzzy nano $\delta\beta$-normal spaces and strongly Pythagorean fuzzy nano $\delta\beta$-normal spaces.

In this paper, we introduce Pythagorean fuzzy nano normal spaces, including their variantsâ€"δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$-normal spacesâ€"as well as their corresponding strongly Pythagorean fuzzy nano δ-type normal spaces. We explore and analyze their fundamental properties within the framework of Pythagorean fuzzy nano topological spaces ($\mathcal{PFN}ts$'s).

Preliminaries

Definition 2.1 *[41] Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A: X \to [0,1]$. That is:*

$$\mu_A(x) = \begin{cases} 1, & \text{if} \quad x \in X \\ 0, & \text{if} \quad x \notin X \\ (0,1) & \text{if } x \text{ ispartlyin } X. \end{cases}$$

Alternatively, a fuzzy set A in X is an object having the form $A = \{< x, \mu_A(x) > | x \in X\}$ or $A = \left\{\left\langle \frac{\mu_A(x)}{x} \right\rangle | x \in X\right\}$, where the function $\mu_A(x): X \to [0,1]$ defines the degree of membership of the element, $x \in X$.

The more x belongs to A, where the grades 1 and 0 denote full membership and full nonmembership, respectively, the closer the membership value $\mu_A(x)$ is to 1, a fuzzy set is a group of items with varying degrees of membership, or graded membership.

The traditional concept of a set is expanded upon by fuzzy sets. In classical set theory, an element's membership in a set is evaluated in binary terms based on a bivalent condition; it either belongs to the set or it doesn't.

Crisp sets are what fuzzy set theory refers to as classical bivalent sets. Since the indicator function of classical sets is a specific instance of the membership functions of fuzzy sets, fuzzy sets are generalized classical sets. If the latter only accept values 0 or 1. With the use of a membership function valued in the real unit interval, fuzzy sets theory enables the incremental evaluation of an element's membership in a set $[0,1]$. Let's look at two instances:

(i) every employee of XYZ who is taller than $1.8m$; (ii) every employee of XYZ who is tall.

In the first example, a classical set and a universe (all XYZ employees) are separated into members (those over $1.8m$) and nonmembers using a membership rule. Because some employees are obviously in the set and some are definitely not, but some are borderline, the second example is a fuzzy set.

The membership function, μ, makes this distinction between the ins, the outs, and the borderline more precise. Using our second example once more, if x is a member of the universe and A is the fuzzy set of all tall employees, X (i.e., all employees), then $\mu_A(x)$ would be $\mu_A(x) = 1$ if x is unquestionably tall, or $\mu_A(x) = 0$ if x is non-tall, or $0 < \mu_A(x) < 1$ for borderline circumstances.

Definition 2.2 *[9, 10, 11, 12] Let a nonempty set X be fixed. An IFS A in X is an object having the form:* $A = \{< x, \mu_A(x), \lambda_A(x) > | x \in X\}$ *or* $A = \left\{\left\langle\frac{\mu_A(x), \lambda_A(x)}{x}\right\rangle | x \in X\right\}$, *where the functions* $\mu_A(x): X \to [0,1]$ *and* $\lambda_A(x): X \to [0,1]$ *define the degree of membership and the degree of nonmembership, respectively, of the element* $x \in X$ *to* A, *which is a subset of* X, *and for every* $x \in X$: $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$. *For each* A *in* X: $\pi_A(x) = 1 - \mu_A(x) - \lambda_A(x)$ *is the intuitionistic fuzzy set index or hesitation margin of* x *in* X. *The hesitation margin* $\pi_A(x)$ *is the degree of nondeterminacy of* $x \in X$ *to the set* A *and* $\pi_A(x) \in [0,1]$. *The hesitation margin is the function that expresses lack of knowledge of whether* $x \in X$ *or* $x \notin X$. *Thus:* $\mu_A(x) + \lambda_A(x) + \pi_A(x) = 1$.

Example 2.1 *Let* $X = \{x, y, z\}$ *be a fixed universe of discourse and* $A = \left\{\left\langle\frac{0.6,0.1}{x}\right\rangle, \left\langle\frac{0.8,0.1}{y}\right\rangle, \left\langle\frac{0.5,0.3}{z}\right\rangle\right\}$, *be the intuitionistic fuzzy set in* X. *The hesitation margins of the elements* x, y, z *to* A *are as follows:* $\pi_A(x) = 0.3, \pi_A(y) = 0.1$ *and* $\pi_A(z) = 0.2$.

Definition 2.3 *[37, 38, 40] Let a non empty set X be a universal set. Then, a Pythagorean fuzzy set A, which is a set of ordered pairs over X, is defined by the following:* $A = \{< x, \mu_A(x), \lambda_A(x) | x \in X\}$ *or* $A = \left\{\left\langle\frac{\mu_A(x), \lambda_A(x)}{x}\right\rangle | x \in X\right\}$, *where the functions* $\mu_A(x): X \to [0,1]$ *and* $\lambda_A(x): X \to [0,1]$ *define the degree of membership and the degree of nonmembership, respectively, of the element* $x \in X$ *to* A, *which is a subset of* X, *and for every* $x \in X$, $0 \leq (\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$. *Supposing* $(\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$, *then there is a degree of indeterminacy of* $x \in X$ *to* A *defined by* $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\lambda_A(x))^2]}$ *and* $\pi_A(x) \in [0,1]$. *In what follows,* $(\mu_A(x))^2 + (\lambda_A(x))^2 + (\pi_A(x))^2 = 1$. *Otherwise,* $\pi_A(x) = 0$ *whenever* $(\mu_A(x))^2 + (\lambda_A(x))^2 = 1$. *We denote the set of all* pfs*'s over* X *by* $pfs(X)$.

Definition 2.4 *[40] Let A and B be pfs's of the forms* $A = \{< a, \mu_A(a), \lambda_A(a) > | a \in X\}$ *and* $B = \{< a, \mu_B(a), \lambda_B(a) > | a \in X\}$. *Then [(i)]*

1. $A \subseteq B$ if and only if $\mu_A(a) \leq \mu_B(a)$ and $\lambda_A(a) \geq \lambda_B(a)$ for all $a \in X$.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $\bar{A}^c = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$.
4. $A \cap B = \{< a, \mu_A(a) \wedge \mu_B(a), \lambda_A(a) \vee \lambda_B(a) > | a \in X\}$.
5. $A \cup B = \{< a, \mu_A(a) \vee \mu_B(a), \lambda_A(a) \wedge \lambda_B(a) > | a \in X\}$.
6. $0_P = \{< a, 0,1 > | a \in X\}$ and $1_P = \{< a, 1,0 > | a \in X\}$.
7. $\bar{1}_P = 0_P$ and $\bar{0}_P = 1_P$.

Definition 2.5 *[4] Let U be a non-empty set and R be an equivalence relation on U. Let A be a Pythagorean fuzzy set in U with the membership function $\mu_A(x)$ and non membership function $\lambda_A(x)$, $\forall\ x \in U$. The Pythagorean fuzzy nano lower approximation, Pythagorean fuzzy nano upper approximation and Pythagorean fuzzy nano boundary approximation of A in (U, R) denoted by $\underline{\mathcal{PF}\mathfrak{N}}(A), \overline{\mathcal{PF}\mathfrak{N}}(A)$ and $B_{\mathcal{PF}\mathfrak{N}}(A)$ and they are respectively defined as follows: [(i)]*

1. $\underline{\mathcal{PF}\mathfrak{N}}(A) = \left\{\langle x, \mu_{\underline{R}(A)}(x), \lambda_{\overline{R}(A)}(x)\rangle / y \in [x]_R, x \in U\right\}$
2. $\overline{\mathcal{PF}\mathfrak{N}}(A) = \left\{\langle x, \mu_{\overline{R}(A)}(x), \lambda_{\underline{R}(A)}(x)\rangle / y \in [x]_R, x \in U\right\}$
3. $B_{\mathcal{PF}\mathfrak{N}}(A) = \overline{\mathcal{PF}\mathfrak{N}}(A) - \underline{\mathcal{PF}\mathfrak{N}}(A)$

where $\mu_{\underline{R}(A)}(x) = \wedge_{y \in [x]_R} \mu_A(y)$

$$\lambda_{\underline{R}(A)}(x) = \wedge_{y \in [x]_R} \lambda_A(y),$$
$$\mu_{\overline{R}(A)}(x) = \vee_{y \in [x]_R} \mu_A(y),$$
$$\lambda_{\overline{R}(A)}(x) = \vee_{y \in [x]_R} \lambda_A(y).$$

Definition 2.6 *[4] Let* U *be an universe of discourse,* R *be an equivalence relation on* U *and* A *be a Pythagorean fuzzy set in* U *and if the collection* $\tau_{\mathcal{P}}(A) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFN}}(A), \overline{\mathcal{PFN}}(A), B_{\mathcal{PFN}}(A)\}$ *forms a topology then it is said to be a Pythagorean fuzzy nano topology. We call* $(U, \tau_{\mathcal{P}}(A))$ *(or simply* U*) as the Pythagorean fuzzy nano topological space. The elements of* $\tau_{\mathcal{P}}(A)$ *are called Pythagorean fuzzy nano open (briefly,* $\mathcal{PFN}o$*) sets.*

Remark 2.1 *[4]* $[\tau_{\mathcal{P}}(A)]^c$ *is called the dual fuzzy nano topology of* $\tau_{\mathcal{P}}(A)$*. In short,* $\mathcal{PFN}c$ *sets are Pythagorean fuzzy nano closed elements of* $[\tau_{\mathcal{P}}(A)]^c$*. Therefore, we see that if and only if* $1_P - G$ *is Pythagorean fuzzy nano open in* $\tau_{\mathcal{P}}(A)$*, then a Pythagorean fuzzy set* G *of* U *is pythagorean fuzzy nano closed in* $\tau_{\mathcal{P}}(A)$*.*

Definition 2.7 *[4, 5] Let* $(U, \tau_{\mathcal{P}}(A))$ *be a* $\mathcal{PFN}ts$ *with respect to* A *where* A *is a* pfs *of* U*. Let* S *be a* pfs *of* U*. Then the Pythagorean fuzzy nano [(i)]*

1. interior of S (briefly, $\mathcal{PFN}int(S)$) is defined by $\mathcal{PFN}int(S) = \cup\{I : I \subseteq S \ \& \ I \text{ is a } \mathcal{PFN}o \text{ set in } U\}$.
2. closure of S (briefly, $\mathcal{PFN}cl(S)$) is defined by $\mathcal{PFN}cl(S) = \cap\{A : S \subseteq A \ \& \ A \text{ is a } \mathcal{PFN}c \text{ set in } U\}$.
3. regular open (briefly, $\mathcal{PFN}ro$) set if $S = \mathcal{PFN}int(\mathcal{PFN}cl(S))$.
4. regular closed (briefly, $\mathcal{PFN}rc$) set if $S = \mathcal{PFN}cl(\mathcal{PFN}int(S))$.

Definition 2.8 *[6] Let* $(U_1, \tau_P(A_1))$ *and* $(U_2, \tau_P(A_2))$ *be two* $\mathcal{PFN}ts$*'s. Then a function* $h_P : U_1 \to U_2$ *is said to be a Pythagorean fuzzy nano continuous (briefly,* $\mathcal{PFN}Cts$*) function if* $h_P^{-1}(G)$ *is* $\mathcal{PFN}o$ *set in* U_1 *for all* $\mathcal{PFN}o$ *set* G *in* U_2*.*

Definition 2.9 *[14] Let* $(U, \tau_{\mathcal{P}}(A))$ *be an* $\mathcal{PFN}ts$ *and* $A = \{< a, \mu_A(a), \lambda_A(a) > | a \in X\}$ *be an* pfs *in* X*. Then the* δ*-interior and the* δ*-closure of* A *are denoted by* $\mathcal{PFN}\delta int(A)$ *and* $\mathcal{PFN}\delta cl(A)$ *and are defined as follows.* $\mathcal{PFN}\delta int(A) = \cup\{G | G$ *is an* $\mathcal{PFN}ros$ *and* $G \subseteq A\}$ *and* $\mathcal{PFN}\delta cl(A) = \cap\{K | K$ *is an* $\mathcal{PFN}rcs$ *and* $A \subseteq K\}$

Definition 2.10 *[14] Let* $(U, \tau_{\mathcal{P}}(A))$ *be an* $\mathcal{PFN}ts$ *and* $A = \{< a, \mu_A(a), \lambda_A(a) > | a \in X\}$ *be an* pfs *in* X*. A set* A *is said to be* $\mathcal{PFN}$ *[(i)]*

1. δ-open set (briefly, $\mathcal{PFN}\delta os$) if $A = \mathcal{PFN}\delta int(A)$,
2. δ-pre open set (briefly, $\mathcal{PFN}\delta Pos$) if $A \subseteq \mathcal{PFN}int(\mathcal{PFN}\delta cl(A))$.
3. δ-semi open set (briefly, $\mathcal{PFN}\delta Sos$) if $A \subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(A))$.
4. δ - α open set or a-open set (briefly, $\mathcal{PFN}\delta\alpha os$ or $\mathcal{PFN}aos$) if $A \subseteq \mathcal{PFN}int(\mathcal{PFN}cl(\mathcal{PFN}\delta int(A)))$.
5. δ - β open set or e^*-open set (briefly, $\mathcal{PFN}\delta\beta os$ or $\mathcal{PFN}e^*os$) if $A \subseteq \mathcal{PFN}cl(\mathcal{PFN}int(\mathcal{PFN}\delta cl(A)))$.
6. δ (resp. δ-pre, δ-semi, δ - α and δ - β) dense if $\mathcal{PFN}\delta cl(A)$ (resp. $\mathcal{PFN}\delta pcl(A), \mathcal{PFN}\delta Scl(A), \mathcal{PFN}\delta\alpha cl(A)$ and $\mathcal{PFN}\delta\beta cl(A)) = 1_P$.

The complement of an $\mathcal{PFN}\delta os$ (resp. $\mathcal{PFN}\delta Pos, \mathcal{PFN}\delta Sos, \mathcal{PFN}\delta\alpha os$ and $\mathcal{PFN}\delta\beta os$) is called an $\mathcal{PFN}\delta$ (resp. $\mathcal{PFN}\delta P, \mathcal{PFN}\delta S, \mathcal{PFN}\delta\alpha$ and $\mathcal{PFN}\delta\beta$) closed set (briefly, $\mathcal{PFN}\delta cs$ (resp. $\mathcal{PFN}\delta Pcs, \mathcal{PFN}\delta Scs, \mathcal{PFN}\delta\alpha cs$ and $\mathcal{PFN}\delta\beta cs$ in X.

The family of all $\mathcal{PFN}\delta os$ (resp. $\mathcal{PFN}\delta cs, \mathcal{PFN}\delta Pos, \mathcal{PFN}\delta Pcs, \mathcal{PFN}\delta Sos, \mathcal{PFN}\delta Scs, \mathcal{PFN}\delta\alpha os, \mathcal{PFN}\delta\alpha cs,$ $\mathcal{PFN}\delta\beta os$ and $\mathcal{PFN}\delta\beta cs$) of X is denoted by $\mathcal{PFN}\delta OS(X)$, (resp. $\mathcal{PFN}\delta CS(X), \mathcal{PFN}\delta POS(X), \mathcal{PFN}\delta PCS(X),$ $\mathcal{PFN}\delta SOS(X), \mathcal{PFN}\delta SCS(X),$ $\mathcal{PFN}\delta\alpha OS(X), \mathcal{PFN}\delta\alpha CS(X),$ $\mathcal{PFN}\delta\beta OS(X)$ and $\mathcal{PFN}\delta\beta CS(X)$).

Definition 2.11 *[14] Let* $(U, \tau_{\mathcal{P}}(A))$ *be an* $\mathcal{PFN}ts$ *and* $A = \{< a, \mu_A(a), \lambda_A(a) > | a \in X\}$ *be an* pfs *in* X*. Then the* $\mathcal{PFN}\delta$*-pre (resp.* $\mathcal{PFN}\delta$*-semi,* $\mathcal{PFN}\delta\alpha$ *and* $\mathcal{PFN}\delta\beta$*)-interior and the* $\mathcal{PFN}\delta$*-pre (resp.* $\mathcal{PFN}\delta$*-semi,* $\mathcal{PFN}\delta\alpha$ *and* $\mathcal{PFN}\delta\beta$*)-closure of* A *are denoted by* $\mathcal{PFN}\delta Pint(A)$ *(resp.* $\mathcal{PFN}\delta Sint(A)$*,* $\mathcal{PFN}\delta\alpha int(A)$ *and* $\mathcal{PFN}\delta\beta int(A)$*) and the* $\mathcal{PFN}\delta Pcl(A)$ *(resp.* $\mathcal{PFN}\delta Scl(A), \mathcal{PFN}\delta\alpha cl(A)$ *and*

$\mathcal{PFN}\delta\beta cl(A)$ *and are defined as follows:*
$\mathcal{PFN}\delta\mathcal{P}int(A)$ (resp. $\mathcal{PFN}\delta\mathcal{S}int(A), \mathcal{PFN}\delta\alpha int(A)$ and $\mathcal{PFN}\delta\beta int(A)$) $=\cup\{G|G$ in a $\mathcal{PFN}\delta\mathcal{P}os$ (resp. $\mathcal{PFN}\delta\mathcal{S}os,$ $\mathcal{PFN}\delta\ \alpha os$ and $\mathcal{PFN}\delta\beta os$)and $G\subseteq A\}$ and $\mathcal{PFN}\delta\mathcal{P}cl(A)$ (resp. $\mathcal{PFN}\delta\mathcal{S}cl(A), \mathcal{PFN}\delta\alpha cl(A)$ and $\mathcal{PFN}\delta\beta cl(A)$) $=\cap\{K|K$ is an $\mathcal{PFN}\delta\mathcal{P}cs$ (resp. $\mathcal{PFN}\delta\mathcal{S}cs, \mathcal{PFN}\delta\alpha cs,$ $\mathcal{PFN}\delta\beta cs$) and $A\subseteq K\}$.

Definition 2.12 *[14] Let $(U_1,\tau_{\mathcal{P}}(A_1))$ and $(U_2,\tau_{\mathcal{P}}(A_2))$ be any two $\mathcal{PFN}ts$'s. A mapping $h_P:(U_1,\tau_{\mathcal{P}}(A_1))\to(U_2,\tau_{\mathcal{P}}(A_2))$ is said to be a Pythagorean fuzzy*
[(i)]

1. continuous (briefly, $\mathcal{PFN}Cts$), if the inverse image of every $\mathcal{PFN}os$ in $(U_2,\tau_{\mathcal{P}}(A_2))$ is a $\mathcal{PFN}os$ in $(U_1,\tau_{\mathcal{P}}(A_1))$.

2. δ-continuous (briefly, $\mathcal{PFN}\delta Cts$), if the inverse image of every $\mathcal{PFN}os$ in $(U_2,\tau_{\mathcal{P}}(A_2))$ is a $\mathcal{PFN}\delta os$ in $(U_1,\tau_{\mathcal{P}}(A_1))$.

3. δP-continuous (briefly, $\mathcal{PFN}\delta\mathcal{P}Cts$), if the inverse image of every $\mathcal{PFN}os$ in $(U_2,\tau_{\mathcal{P}}(A_2))$ is a $\mathcal{PFN}\delta\mathcal{P}os$ in $(U_1,\tau_{\mathcal{P}}(A_1))$.

4. δS-continuous (briefly, $\mathcal{PFN}\delta\mathcal{S}Cts$), if the inverse image of every $\mathcal{PFN}os$ in $(U_2,\tau_{\mathcal{P}}(A_2))$ is a $\mathcal{PFN}\delta\mathcal{S}os$ in $(U_1,\tau_{\mathcal{P}}(A_1))$.

5. $\delta\alpha$-continuous (briefly, $\mathcal{PFN}\delta\alpha Cts$), if the inverse image of every $\mathcal{PFN}os$ in $(U_2,\tau_{\mathcal{P}}(A_2))$ is a $\mathcal{PFN}\delta\alpha os$ in $(U_1,\tau_{\mathcal{P}}(A_1))$.

6. $\delta\beta$-continuous (briefly, $\mathcal{PFN}\delta\beta Cts$), if the inverse image of every $\mathcal{PFN}os$ in $(U_2,\tau_{\mathcal{P}}(A_2))$ is a $\mathcal{PFN}\delta\beta os$ in $(U_1,\tau_{\mathcal{P}}(A_1))$.

Definition 2.13 *[17] Let $(U_1,\tau_{\mathcal{P}}(A_1))$ & $(U_2,\tau_{\mathcal{P}}(A_2))$ be a $\mathcal{PFN}ts$'s. A mapping $h_P:(U_1,\tau_{\mathcal{P}}(A_1))\to(U_2,\tau_{\mathcal{P}}(A_2))$ is said to be a Pythagorean fuzzy nano (resp. δ, $\delta\mathcal{S}$ and $\delta\mathcal{P}$)-open map (briefly, $\mathcal{PFN}O$ (resp. $\mathcal{PFN}\delta O$, $\mathcal{PFN}\delta\mathcal{S}O$ and $\mathcal{PFN}\delta\mathcal{P}O$)) if the image of every $\mathcal{PFN}os$ in $(U_1,\tau_{\mathcal{P}}(A_1))$ is a $\mathcal{PFN}os$ (resp. $\mathcal{PFN}\delta os$, $\mathcal{PFN}\delta\mathcal{S}os$ and $\mathcal{PFN}\delta\mathcal{P}os$) in $(U_2,\tau_{\mathcal{P}}(A_2))$.*

Definition 2.14 *[17] Let $(U_1,\tau_{\mathcal{P}}(A_1))$ & $(U_2,\tau_{\mathcal{P}}(A_2))$ be any two $\mathcal{PFN}ts$'s. A mapping $h_P:(U_1,\tau_{\mathcal{P}}(A_1))\to(U_2,\tau_{\mathcal{P}}(A_2))$ is said to be Pythagorean fuzzy nano (resp. δ, $\delta\mathcal{S}$ and $\delta\mathcal{P}$) closed map (briefly, $\mathcal{PFN}C$ (resp. $\mathcal{PFN}\delta C$, $\mathcal{PFN}\delta\mathcal{S}C$ and $\mathcal{PFN}\delta\mathcal{P}C$)) if the image of every $\mathcal{PFN}cs$ in $(U_1,\tau_{\mathcal{P}}(A_1))$ is a $\mathcal{PFN}cs$ (resp. $\mathcal{PFN}\delta cs$, $\mathcal{PFN}\delta\mathcal{S}cs$ and $\mathcal{PFN}\delta\mathcal{P}cs$) in $(U_2,\tau_{\mathcal{P}}(A_2))$.*

Pythagorean fuzzy nano (resp. $\boldsymbol{\delta}$, $\boldsymbol{\delta}$ pre, $\boldsymbol{\delta}$ semi, $\boldsymbol{\delta\alpha}$ and $\boldsymbol{\delta\beta}$) normal spaces

In this section, we introduce Pythagorean fuzzy nano normal spaces and their variants - δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$ - and examine their properties.

Definition 3.1 *Let $(U,\tau_{\mathcal{P}}(A))$ be a $\mathcal{PFN}ts$ is said to be Pythagorean fuzzy (resp. δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) normal (briefly, $\mathcal{PFN}Nor$ (resp. $\mathcal{PFN}\delta Nor$, $\mathcal{PFN}\delta\mathcal{P}Nor$, $\mathcal{PFN}\delta\mathcal{S}Nor$, $\mathcal{PFN}\delta\alpha Nor$ and $\mathcal{PFN}\delta\beta Nor$)) if for any two disjoint $\mathcal{PFN}c$ (resp. $\mathcal{PFN}\delta c$, $\mathcal{PFN}\delta\mathcal{P}c$, $\mathcal{PFN}\delta\mathcal{S}c$, $\mathcal{PFN}\delta\alpha c$ and $\mathcal{PFN}\delta\beta c$) sets A and B, there exist disjoint $\mathcal{PFN}o$ (resp. $\mathcal{PFN}\delta o$, $\mathcal{PFN}\delta\mathcal{P}o$, $\mathcal{PFN}\delta\mathcal{S}o$, $\mathcal{PFN}\delta\alpha o$ and $\mathcal{PFN}\delta\beta o$) sets L and M such that $A\subseteq L$ and $B\subseteq M$.*

Theorem 3.1 *In a $\mathcal{PFN}ts$ $(U,\tau_{\mathcal{P}}(A))$, the following are equivalent: [(i)]*

1. U is $\mathcal{PFN}\delta\beta Nor$.

2. For every $\mathcal{PFN}\delta\beta c$ set A in U and every $\mathcal{PFN}\delta\beta o$ set L containing A, there exists a $\mathcal{PFN}\delta\beta o$ set M containing A such that $\mathcal{PFN}cl(M) \subseteq L$.
3. For each pair of disjoint $\mathcal{PFN}\delta\beta c$ sets A and B in X, there exists a $\mathcal{PFN}\delta\beta o$ set L containing A such that $\mathcal{PFN}cl(L) \cap B = 0_P$.
4. For each pair of disjoint $\mathcal{PFN}\delta\beta c$ sets A and B in X, there exist $\mathcal{PFN}\delta\beta o$ sets L and M containing A and B respectively such that $\mathcal{PFN}cl(L) \cap \mathcal{PFN}cl(M) = 0_P$.

Proof. (i) ⇒ (ii): Let L be a $\mathcal{PFN}\delta\beta o$ set containing the $\mathcal{PFN}\delta\beta c$ set A. Then $B = L^c$ is a $\mathcal{PFN}\delta\beta c$ set disjoint from A. Since U is $\mathcal{PFN}\delta\beta Nor$, there exist disjoint $\mathcal{PFN}\delta\beta o$ sets M and W containing A and B respectively. Then $\mathcal{PFN}cl(M)$ is disjoint from B. Since if $y_\beta \in B$, the set W is a $\mathcal{PFN}\delta\beta o$ set containing y_β disjoint from M. Hence $\mathcal{PFN}cl(M) \subseteq L$.
(ii) ⇒ (iii): Let A and B be disjoint $\mathcal{PFN}\delta\beta c$ sets in X. Then B^c is a $\mathcal{PFN}\delta\beta o$ set containing A. By (ii), there exists a $\mathcal{PFN}\delta\beta o$ set L containing A such that $\mathcal{PFN}cl(L) \subseteq B^c$. Hence $\mathcal{PFN}cl(L) \cap B = 0_P$. This proves (iii).
(iii) ⇒ (iv): Let A and B be disjoint $\mathcal{PFN}\delta\beta c$ sets in X. Then, by (iii), there exists a $\mathcal{PFN}\delta\beta o$ set L containing A such that $\mathcal{PFN}cl(L) \cap B = 0_P$. Since $\mathcal{PFN}cl(L)$ is $\mathcal{PFN}\delta\beta c$ set, B and $\mathcal{PFN}cl(L)$ are disjoint $\mathcal{PFN}\delta\beta c$ sets in X. Again by (iii), there exists a $\mathcal{PFN}\delta\beta o$ set M containing B such that $\mathcal{PFN}cl(L) \cap \mathcal{PFN}cl(M) = 0_P$. This proves (iv).
(iv) ⇒ (i): Let A and B be the disjoint $\mathcal{PFN}\delta\beta c$ sets in X. By (iv), there exist $\mathcal{PFN}\delta\beta o$ sets L and M containing A and B respectively such that $\mathcal{PFN}cl(L) \cap \mathcal{PFN}cl(M) = 0_P$. Since $L \cap M \subseteq \mathcal{PFN}cl(L) \cap \mathcal{PFN}cl(M)$, L and M are disjoint $\mathcal{PFN}\delta\beta o$ sets containing A and B respectively. Thus X is $\mathcal{PFN}\delta\beta Nor$. width 0.22 true cm height 0.22 true cm depth 0pt

Theorem 3.2 *Let $(U, \tau_{\mathcal{P}}(A))$ be a $\mathcal{PFN}ts$ is $\mathcal{PFN}\delta\beta Nor$ if and only if for every $\mathcal{PFN}\delta\beta c$ set F and $\mathcal{PFN}\delta\beta o$ set G containing F, there exists a $\mathcal{PFN}\delta\beta o$ set M such that $F \subseteq M \subseteq \mathcal{PFN}cl(M) \subseteq G$.*

Proof. Let $(U, \tau_{\mathcal{P}}(A))$ be $\mathcal{PFN}\delta\beta Nor$. Let F be a $\mathcal{PFN}\delta\beta c$ set and let G be a $\mathcal{PFN}\delta\beta o$ set containing F. Then F and G^c are disjoint $\mathcal{PFN}\delta\beta c$ sets. Since X is $\mathcal{PFN}\delta\beta Nor$, there exist disjoint $\mathcal{PFN}\delta\beta o$ sets M_1 and M_2 such that $F \subseteq M_1$ and $G^c \subseteq M_2$. Thus $F \subseteq M_1 \subseteq M_2^c \subseteq G$. Since M_2^c is $\mathcal{PFN}\delta\beta c$ set, so $\mathcal{PFN}cl(M) \subseteq \mathcal{PFN}cl(M_2^c) = M_2^c \subseteq G$. Take $M = M_1$.This implies that $F \subseteq M \subseteq \mathcal{PFN}cl(M) \subseteq G$.
Conversely, suppose the condition holds. Let H_1 and H_2 be two disjoint $\mathcal{PFN}\delta\beta c$ sets in U. Then H_2^c is a $\mathcal{PFN}\delta\beta o$ set containing H_1. By assumption, there exists a $\mathcal{PFN}\delta\beta o$ set M such that $H_1 \subseteq M \subseteq \mathcal{PFN}cl(M) \subseteq H_2^c$ Since M is $\mathcal{PFN}\delta\beta o$ and $\mathcal{PFN}cl(M)$ is $\mathcal{PFN}\delta\beta c$. Then $(\mathcal{PFN}cl(M))^c$ is $\mathcal{PFN}\delta\beta o$. Now $\mathcal{PFN}cl(M) \subseteq H_2^c$ implies that $H_2 \subseteq (\mathcal{PFN}cl(M))^c$ Also $M \cap (\mathcal{PFN}cl(M))^c \subseteq \mathcal{PFN}cl(M) \cap (\mathcal{PFN}cl(M))^c = 0_P$. That is M and $(\mathcal{PFN}cl(M))^c$ are disjoint $\mathcal{PFN}\delta\beta o$ sets containing H_1 and H_2 respectively. This shows that $(U, \tau_{\mathcal{P}}(A))$ is $\mathcal{PFN}\delta\beta Nor$. width 0.22 true cm height 0.22 true cm depth 0pt

Theorem 3.3 *For a $\mathcal{PFN}ts$ $(U, \tau_{\mathcal{P}}(A))$, then the following are equivalent: [(i)]*

1. U is $\mathcal{PFN}\delta\beta Nor$.
2. For any two $\mathcal{PFN}\delta\beta o$ sets L and M whose union is 1_P, there exist $\mathcal{PFN}\delta\beta c$ subsets A_0 of L and B_0 of M whose union is also 1_P.

Proof. (i) ⇒ (ii): Let L and M be two $\mathcal{PFN}\delta\beta o$ sets in a $\mathcal{PFN}\delta\beta Nor$ space U such that $1_P = L \cup M$. Then L^c, M^c are disjoint $\mathcal{PFN}\delta\beta c$ sets. Since X is $\mathcal{PFN}\delta\beta Nor$, then there exist disjoint $\mathcal{PFN}\delta\beta o$ sets G_1 and G_2 such that $L^c \subseteq G_1$ and $M^c \subseteq G_2$. Let $A_0 = G_1^c$ and $B_0 = G_2^c$. Then A_0 and B_0 are $\mathcal{PFN}\delta\beta c$ subsets of L and M respectively such that $A_0 \cup B_0 = 1_P$. This proves (ii).
(ii) ⇒ (i): Let A_0 and B_0 be disjoint $\mathcal{PFN}\delta\beta c$ sets in U. Then A_0^c and B_0^c are $\mathcal{PFN}\delta\beta o$ sets whose union is 1_P. By (ii), there exists $\mathcal{PFN}\delta\beta c$ sets F_1 and F_2 such that $F_1 \subseteq A_0^c, F_2 \subseteq B_0^c$ and $F_1 \cup F_2 = 1_P$. Then F_1^c and F_2^c are disjoint $\mathcal{PFN}\delta\beta o$ sets containing A_0 and B_0 respectively. Therefore U is

$\mathcal{PF}\mathfrak{N}\delta\beta Nor$. width 0.22 true cm height 0.22 true cm depth 0pt

Theorem 3.4 *Let $h_P: (U_1, \tau_{\mathcal{P}}(A_1)) \to (U_2, \tau_{\mathcal{P}}(A_2))$ be a function. [(i)]*

1. If h_P is injective, $\mathcal{PF}\mathfrak{N}\delta\beta Irr$, $\mathcal{PF}\mathfrak{N}\delta\beta O$ and U_1 is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$ then U_2 is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$.
2. If h_P is $\mathcal{PF}\mathfrak{N}\delta\beta Irr$, $\mathcal{PF}\mathfrak{N}\delta\beta C$ and U_2 is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$ then U_1 is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$.

Proof. (i) Suppose U_1 is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$. Let A and B be disjoint $\mathcal{PF}\mathfrak{N}\delta\beta c$ sets in U_2. Since h_P is $\mathcal{PF}\mathfrak{N}\delta\beta Irr$, $h_P^{-1}(A)$ and $h_P^{-1}(B)$ are $\mathcal{PF}\mathfrak{N}\delta\beta c$ in U_1. Since U_1 is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$, there exist disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets L and M in X such that $h_P^{-1}(A) \subseteq L$ and $h_P^{-1}(B) \subseteq M$. Now $h_P^{-1}(A) \subseteq L \Rightarrow A \subseteq h_P(L)$ and $h_P^{-1}(B) \subseteq M \Rightarrow B \subseteq h_P(M)$. Since h_P is a $\mathcal{PF}\mathfrak{N}\delta\beta O$ map, $h_P(L)$ and $h_P(M)$ are $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets in U_2. Also $L \cap M = 0_P \Rightarrow h_P(L \cap M) = 0_P$ and h_P is injective, then $h_P(L) \cap h_P(M) = 0_P$. Thus $h_P(L)$ and $h_P(M)$ are disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets in U_2 containing A and B respectively. Thus, U_2 is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$.
(ii) Suppose U_2 is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$. Let A and B be disjoint $\mathcal{PF}\mathfrak{N}\delta\beta c$ sets in U_1. Since h_P is $\mathcal{PF}\mathfrak{N}\delta\beta Irr$ and $\mathcal{PF}\mathfrak{N}\delta\beta C$, $h_P(A)$ and $h_P(B)$ are $\mathcal{PF}\mathfrak{N}\delta\beta c$ sets in U_2. Since U_2 is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$, there exist disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets L and M in U_2 such that $h_P(A) \subseteq L$ and $h_P(B) \subseteq M$. That is $A \subseteq h_P^{-1}(L)$ and $B \subseteq h_P^{-1}(M)$. Since h_P is $\mathcal{PF}\mathfrak{N}\delta\beta Irr$, $h_P^{-1}(L)$ and $h_P^{-1}(M)$ are disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets such that $A \subseteq h_P^{-1}(L)$ and $B \subseteq h_P^{-1}(M)$. Thus U_1 is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$. width 0.22 true cm height 0.22 true cm depth 0pt

Theorem 3.5 *If given a pair of disjoint $\mathcal{PF}\mathfrak{N}\delta\beta c$ sets A_0, B_0 of $(U, \tau_{\mathcal{P}}(A))$, there is $\mathcal{PF}\mathfrak{N}\delta\beta Cts$ function h_P such that $h_P(A_0) = 0_P$ and $h_P(B_0) = 1_P$, then $(U, \tau_{\mathcal{P}}(A))$ is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$.*

Proof. Let $(U, \tau_{\mathcal{P}}(A))$ be a $\mathcal{PF}\mathfrak{N}ts$. Suppose for any pair of disjoint $\mathcal{PF}\mathfrak{N}\delta\beta c$ sets A_0, B_0 in U, there exists a $\mathcal{PF}\mathfrak{N}\delta\beta Cts$ map h_P such that $h_P(A_0) = 0_P$ and $h_P(B_0) = 1_P$. Let E and F be disjoint $\mathcal{PF}\mathfrak{N}\delta\beta c$ sets in U. Let G and H be disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets. Since h_P is $\mathcal{PF}\mathfrak{N}\delta\beta Cts$, $h_P^{-1}(G)$ and $h_P^{-1}(H)$ are $\mathcal{PF}\mathfrak{N}\delta\beta o$ in X. By our assumption, $h_P(E) = 0_P$ and $h_P(F) = 1_P$. Now $h_P(E) = 0_P$ implies $h_P^{-1}(h_P(E)) \subseteq h_P^{-1}(0_P) \Rightarrow E \subseteq h_P^{-1}(h_P(E)) \subseteq h_P^{-1}(0_P) \Rightarrow E \subseteq h_P^{-1}(0_P)$. Similarly $F \subseteq h_P^{-1}(1_P)$. This implies that $E \subseteq h_P^{-1}(0_P) \subseteq h_P^{-1}(G)$. Then $F \subseteq h_P^{-1}(1_P) \subseteq h_P^{-1}(H)$. Further, $h_P^{-1}(G) \cap h_P^{-1}(H) = h_P^{-1}(G \cap H) = h_P^{-1}(0_P) = 0_P$. So, we have a pair of disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets, $h_P^{-1}(G), h_P^{-1}(H) \subseteq 1_P$ such that $E \subseteq h_P^{-1}(G)$ and $F \subseteq h_P^{-1}(H)$. This proves that (X, Γ_P) is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$. width 0.22 true cm height 0.22 true cm depth 0pt

Theorem 3.6 *Let $h_P: (U_1, \tau_{\mathcal{P}}(A_1)) \to (U_2, \tau_{\mathcal{P}}(A_2))$ be a function. If h_P is a $\mathcal{PF}\mathfrak{N}Cts$, $\mathcal{PF}\mathfrak{N}\delta\beta O$ bijection of a $\mathcal{PF}\mathfrak{N}Nor$ space U_1 into a space U_2 and if every $\mathcal{PF}\mathfrak{N}\delta\beta c$ set in U_2 is $\mathcal{PF}\mathfrak{N}c$ set, then U_2 is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$.*

Proof. Let A and B be $\mathcal{PF}\mathfrak{N}\delta\beta c$ sets in U_2. Then by assumption, B is $\mathcal{PF}\mathfrak{N}c$ set in U_2. Since h_P is a $\mathcal{PF}\mathfrak{N}Cts$ bijection, $h_P^{-1}(A)$ and $h_P^{-1}(B)$ is a $\mathcal{PF}\mathfrak{N}c$ set in U_1. Since U_1 is $\mathcal{PF}\mathfrak{N}Nor$, there exist disjoint $\mathcal{PF}\mathfrak{N}o$ sets L_1 and L_2 in U such that $h_P^{-1}(A) \subseteq L_1$ and $h_P^{-1}(B) \subseteq L_2$. Since h_P is $\mathcal{PF}\mathfrak{N}\delta\beta O$, $h_P(L_1)$ and $h_P(L_2)$ are disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets in U_2 containing A and B respectively. Hence U_2 is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$. width 0.22 true cm height 0.22 true cm depth 0pt

Remark 3.1 *Theorems 3.1, 3.2, 3.3, 3.4, 3.5 & 3.6 are also holds for $\mathcal{PF}\mathfrak{N}o$, $\mathcal{PF}\mathfrak{N}\delta o$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}o$, $\mathcal{PF}\mathfrak{N}\delta\alpha o$ and $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets.*

Strongly Pythagorean fuzzy nano (resp. $\boldsymbol{\delta}$, $\boldsymbol{\delta}$ pre, $\boldsymbol{\delta}$ semi, $\boldsymbol{\delta\alpha}$ and $\boldsymbol{\delta\beta}$) normal spaces

In this section, we introduce strongly Pythagorean fuzzy nano δ normal spaces and their variants δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$-and explore their properties.

Definition 4.1 A $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_{\mathcal{P}}(A))$ *is said to be strongly Pythagorean fuzzy δ (resp. $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) normal (briefly, $St\mathcal{PF}\mathfrak{N}\delta Nor$ (resp. $St\mathcal{PF}\mathfrak{N}\delta\mathcal{P}Nor$, $St\mathcal{PF}\mathfrak{N}\delta\mathcal{S}Nor$, $St\mathcal{PF}\mathfrak{N}\delta\alpha Nor$ and*

$St\mathcal{PF}\mathfrak{N}\delta\beta Nor$)) if for every pair of disjoint $\mathcal{PF}\mathfrak{N}c$ sets A and B in U, there are disjoint $\mathcal{PF}\mathfrak{N}o$ (resp. $\mathcal{PF}\mathfrak{N}\delta o$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}o$, $\mathcal{PF}\mathfrak{N}\delta\alpha o$ and $\mathcal{PF}\mathfrak{N}\delta\beta o$) sets L and M in U containing A and B respectively.

Theorem 4.1 *Let $(U, \tau_{\mathcal{P}}(A))$ be a $\mathcal{PF}\mathfrak{N}ts$. Every $\mathcal{PF}\mathfrak{N}\delta\beta Nor$ space is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$.*

Proof. Suppose U is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$. Let A_0 and B_0 be disjoint $\mathcal{PF}\mathfrak{N}c$ sets in U. Then A_0 and B_0 are $\mathcal{PF}\mathfrak{N}\delta\beta cs$'s in U. Since U is $\mathcal{PF}\mathfrak{N}\delta\beta Nor$, there exist disjoint $\mathcal{PF}\mathfrak{N}o$ sets L and M containing A_0 and B_0 respectively. Since, every $\mathcal{PF}\mathfrak{N}o$ is $\mathcal{PF}\mathfrak{N}\delta\beta o$, L and M are $\mathcal{PF}\mathfrak{N}\delta\beta o$ in U. This implies that U is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$. width 0.22 true cm height 0.22 true cm depth 0pt

Theorem 4.2 *In a $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_{\mathcal{P}}(A))$, the following are equivalent: [(i)]*

1. U is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$.
2. For every $\mathcal{PF}\mathfrak{N}c$ set h_P in U and every $\mathcal{PF}\mathfrak{N}o$ set L containing h_P, there exists a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set M containing h_P such that $\mathcal{PF}\mathfrak{N}\delta\beta cl(M) \subseteq L$.
3. For each pair of disjoint $\mathcal{PF}\mathfrak{N}c$ sets A and B in U, there exists a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set L containing A such that $\mathcal{PF}\mathfrak{N}\delta\beta cl(L) \cap B = 0_P$.

Proof. (i) ⇒ (ii): Let L be a $\mathcal{PF}\mathfrak{N}o$ set containing the $\mathcal{PF}\mathfrak{N}c$ set h_P. Then $H = L^c$ is a $\mathcal{PF}\mathfrak{N}c$ set disjoint from h_P. Since U is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$, there exist disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets M and W containing h_P and H respectively. Then $\mathcal{PF}\mathfrak{N}\delta\beta cl(M)$ is disjoint from H, since if $y_\beta \in H$, the set W is a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set containing y_β disjoint from M. Hence $\mathcal{PF}\mathfrak{N}\delta\beta cl(M) \subseteq L$.
(ii) ⇒ (iii): Let A and B be disjoint $\mathcal{PF}\mathfrak{N}c$ sets in U. Then B^c is a $\mathcal{PF}\mathfrak{N}o$ set containing A. By (ii), there exists a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set L containing A such that $\mathcal{PF}\mathfrak{N}\delta\beta cl(L) \subseteq B^c$. Hence $\mathcal{PF}\mathfrak{N}\delta\beta cl(L) \cap B = 0_P$. This proves (iii).
(iii) ⇒ (i): Let A and B be the disjoint $\mathcal{PF}\mathfrak{N}\delta\beta c$ sets in U. By (iii), there exists a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set L containing A such that $\mathcal{PF}\mathfrak{N}\delta\beta cl(L) \cap B = 0_P$. Take $M = \mathcal{PF}\mathfrak{N}\delta\beta cl(L)^c$. Then L and M are disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets containing A and B respectively. Thus U is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$. width 0.22 true cm height 0.22 true cm depth 0pt

Theorem 4.3 *For a $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_{\mathcal{P}}(A))$, then the following are equivalent: [(i)]*

1. U is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$.
2. For any two $\mathcal{PF}\mathfrak{N}o$ sets L and M whose union is 1_P, there exist $\mathcal{PF}\mathfrak{N}\delta\beta c$ subsets A of L and B of M whose union is also 1_P.

Proof. (i) ⇒ (ii): Let L and M be two $\mathcal{PF}\mathfrak{N}o$ sets in a $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$ space U such that $1_P = L \cup M$. Then L^c, M^c are disjoint $\mathcal{PF}\mathfrak{N}c$ sets. Since U is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$, then there exist disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets G_1 and G_2 such that $L^c \subseteq G_1$ and $M^c \subseteq G_2$. Let $A = G_1^c$ and $B = G_2^c$. Then A and B are $\mathcal{PF}\mathfrak{N}\delta\beta c$ subsets of L and M respectively such that $A \cup B = 1_P$. This proves (ii).
(ii) ⇒ (i): Let A and B be disjoint $\mathcal{PF}\mathfrak{N}c$ sets in U. Then A^c and B^c are $\mathcal{PF}\mathfrak{N}o$ sets whose union is 1_P. By (ii), there exists $\mathcal{PF}\mathfrak{N}\delta\beta c$ sets F_1 and F_2 such that $F_1 \subseteq A^c, F_2 \subseteq B^c$ and $F_1 \cup F_2 = 1_P$. Then F_1^c and F_2^c are disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets containing A and B respectively. Therefore U is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$. width 0.22 true cm height 0.22 true cm depth 0pt

Theorem 4.4 *Let $h_P: (U_1, \tau_{\mathcal{P}}(A_1)) \to (U_2, \tau_{\mathcal{P}}(A_2))$ be a function. [(i)]*

1. If h_P is injective, $\mathcal{PF}\mathfrak{N}Cts$, $\mathcal{PF}\mathfrak{N}\delta\beta O$ and U_1 is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$ then U_2 is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$.
2. If h_P is $\mathcal{PF}\mathfrak{N}\delta\beta Irr$, $\mathcal{PF}\mathfrak{N}\delta\beta O$ and U_2 is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$ then U_1 is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$.

Proof. (i) Suppose U_1 is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$. Let A and B be disjoint $\mathcal{PF}\mathfrak{N}c$ sets in U_2. Since h_P is $\mathcal{PF}\mathfrak{N}Cts$, $h_P^{-1}(A)$ and $h_P^{-1}(B)$ are $\mathcal{PF}\mathfrak{N}c$ in U_1. Since U_1 is $St\mathcal{PF}\mathfrak{N}\delta\beta Nor$, there exist disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets L and M in U_1 such that $h_P^{-1}(A) \subseteq L$ and $h_P^{-1}(B) \subseteq M$. Now $h_P^{-1}(A) \subseteq L \Rightarrow A \subseteq$

$h_P(L)$ and $h_P^{-1}(B) \subseteq M \Rightarrow B \subseteq h_P(M)$. Since h_P is a $\mathcal{PFN}\delta\beta O$ map, $h_P(L)$ and $h_P(M)$ are $\mathcal{PFN}\delta\beta o$ set in U_2. Also $L \cap M = 0_P \Rightarrow h_P(L \cap M) = 0_P$ and h_P is injective, then $h_P(L) \cap h_P(M) = 0_P$. Thus $h_P(L)$ and $h_P(M)$ are disjoint $\mathcal{PFN}\delta\beta o$ sets in U_2 containing A and B respectively. Thus, U_2 is $St\mathcal{PFN}\delta\beta Nor$.
(ii) Suppose U_2 is $\mathcal{PFN}\delta\beta Nor$. Let A and B be disjoint $\mathcal{PFN}c$ sets in U_1. Since h_P is $\mathcal{PFN}\delta\beta Irr$ and $\mathcal{PFN}\delta\beta C$, $h_P(A)$ and $h_P(B)$ are $\mathcal{PFN}\delta\beta c$ sets in U_2. Since U_2 is $\mathcal{PFN}\delta\beta Nor$, there exist disjoint $\mathcal{PFN}\delta\beta o$ sets L and M in $\delta\beta_2$ such that $h_P(A) \subseteq L$ and $h_P(B) \subseteq M$. That is $A \subseteq h_P^{-1}(L)$ and $B \subseteq h_P^{-1}(M)$. Since h_P is $\mathcal{PFN}\delta\beta Irr$, $h_P^{-1}(L)$ and $h_P^{-1}(M)$ are disjoint $\mathcal{PFN}\delta\beta o$ such that $A \subseteq h_P^{-1}(L)$ and $B \subseteq h_P^{-1}(M)$. Thus U_1 is $\mathcal{PFN}\delta\beta Nor$. width 0.22 true cm height 0.22 true cm depth 0pt

Remark 4.1 *Theorems 4.1, 4.2, 4.3 & 4.4 are also holds for $\mathcal{PFN}o$, $\mathcal{PFN}\delta o$, $\mathcal{PFN}\delta\mathcal{P}o$, $\mathcal{PFN}\delta\mathcal{S}o$ and $\mathcal{PFN}\delta\alpha o$ sets.*

CONCLUSION

In this paper, we investigated Pythagorean fuzzy nano (resp. δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$)-normal spaces and strongly Pythagorean fuzzy nano (resp. δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$)-normal spaces using Pythagorean fuzzy nano (resp. δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$)-open and Pythagorean fuzzy nano (resp. δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$)-closed sets. We analyzed the relationships among these newly defined spaces as well as their connections with previously established classes. Furthermore, we examined their fundamental properties and provided characterizations within the framework of Pythagorean fuzzy nano topological structures.

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