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Star – In – Coloring Of Herschel, General Theta And Corona Graphs

T. Namachivayam[1], P. Kasirajan [2], M. Arun kumar [3]

[1,3] Department of Mathematics, Kalaignar Karunanidhi Government Arts College,
(Affiliated to Thiruvalluvar University)
Tiruvannamalai – 606 603, Tamil Nadu, India.
[2] Department of Mathematics,127 - Government Polytechnic College, Nagapadi,
Tiruvannamalai – 606 705, Tamil Nadu, India.
Email: [1]namachisiva1968@gmail.com; [2] kasirajanmaths@gmail.com
[3] drarun4maths@gmail.com

ABSTRACT

A proper coloring of a graph $G = (V, E)$ is a mapping $f: V \to N$ such that if $v_i v_j \epsilon E$ then $f(v_i) \neq f(v_j)$. In this paper we prove that Herschel graph, General Theta graph and Corona graph are star–in–coloring graphs and give the exact value of the Star – in – Chromatic number of these graphs. Also provide the bounds for Star – in – Chromatic number of the corona between cycles and paths.

Key Words: *Star–in–coloring; Herschel graph; General Theta graph; Corona graph.*

AMS Subject Classification: *05C15, 05C20.*

1. INTRODUCTION

Graph coloring is a fundamental concept in graph theory that involves assigning labels, known as colors, to the vertices or edges of a graph subject to specific constraints. The most common form, vertex coloring, requires that no two adjacent vertices share the same color. This ensures that each edge connects vertices of distinct colors. The notion of acyclic coloring was introduced by B. Grunbaum in 1973 [3]. Star coloring is a concept in graph theory that involves assigning colors to the vertices of a graph such that no path of four vertices is bicolored. This means that every path consisting of four vertices must include at least three distinct colors [1,6].In-coloring of a directed graph is a type of vertex coloring where in any path P_3 of length 2 with end vertices share the same color, the edges are directed towards the middle vertex.

Building upon the foundational ideas of star-coloring and in-coloring, S. Sudha, V. Kanniga [7,8] introduced the innovative concept of star–in–coloring for graphs. This new approach integrates the principles of both star-coloring and in-coloring, ensuring that no path of length three (P_4) is bicolored and that in any path of length two (P_3) with identical end vertices, the edges are directed towards the central vertex. Expanding on this concept, A. Sugumaran, P. Kasirajan [9 - 13] conducted further studies to determine the lower and upper bounds of the star–in–chromatic number for various graph classes. This study investigates the concept of star–in–coloring applied to specific Herschel graph, General Theta graph, Corona graph. The analysis begins by outlining essential definitions and observations foundational to graph theory, as detailed in Harary [4].

Definition 1.1[7] The star–in–chromatic number of a graph G, symbolized as $\chi_{si}(G)$, signifies the smallest count of distinct colors necessary to achieve a star–in–coloring of G.

Definition 1.2 [4] A general theta graph $\theta(m, n)$ is a simple graph consisting of two vertices joined by n internal disjoint paths of length m.

Definition 1.3 [2,5] The corona $G_1 o G_2$ of two graphs G_1 with (m_1 vertices and n_1 edges) and G_2 (with m_2 vertices and n_2 edges) is defined as the graph obtained by taking one copy of G_1 and m_1 copies of G_2, and then joining the ith vertex of G_1 with an edge to every vertex in the ith copy of G_2.

2. MAIN RESULTS

Theorem 1. The Herschel graph H_s admits star–in–coloring and its star–in–chromatic number is 5.
Proof: Let H_s be a Herschel graph, which consists of 11 vertices and 18 edges. Let v_0 be the central vertex and $v_i (1 \leq i \leq 10)$ be the remaining vertices of H_s.

Let V be the vertex set of H_s and E be the edge set of H_s. We define a function $f: V \rightarrow \{1, 2, 3, \dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \in E$, as follows:

$$f(v_0) = f(v_9) = 4 \text{ and } f(v_{10}) = 5$$

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod 2 \\ 2, & \text{if } i \equiv 2 \pmod 4 \\ 3, & \text{if } i \equiv 0 \pmod 4 \end{cases}$$

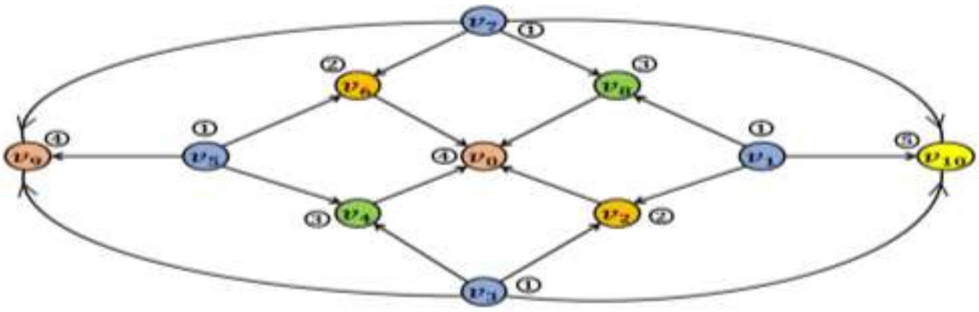

Fig. 1. Star-in-coloring of H_s

In this pattern of coloring, the Herschel graph H_s is star–in–colored and its star–in–chromatic number is $\chi_{si}(H_s) = 5$.

Theorem 2. The general theta graph $\theta(m, n)$ admits star–in–coloring and its star–in–chromatic number is $\chi_{si}[\theta(m, n)] = 3$.

Proof: The graph consists of $(m-1)n + 2$ vertices and mn edges. The vertex set V in $\theta(m, n)$ are partioned into n vertex sets denoted by $V^1, V^2, V^3, \dots, V^n$ where each vertex set consists of $m - 1$ vertices. The vertex set V^j consists of the vertices $v_1^j, v_2^j, \dots, v_{m-1}^j$ for all $1 \leq j \leq n$ and the end vertices are denoted by u_0, v_0.

Let V be the vertex set of $\theta(m, n)$ and E be the edge set of $\theta(m, n)$. We define a function $f: V \rightarrow \{1, 2, 3, \dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \in E$, as the following two cases:

Case 1: $m \equiv 0 \pmod 4$ or $m \equiv 3 \pmod 4$

$$f(u_0) = f(v_0) = 1$$

$$f\left(v_i^j\right) = \begin{cases} 2, & \text{if } i \equiv 1 \pmod 2 \\ 3, & \text{if } i \equiv 2 \pmod 4 \\ 1, & \text{if } i \equiv 0 \pmod 4 \end{cases}$$

Case 2: $m \equiv 1 \pmod 4$ or $m \equiv 2 \pmod 4$

$$f(u_0) = 1, \qquad f(v_0) = 3$$

$$f\left(v_i^j\right) = \begin{cases} 2, & \text{if } i \equiv 1 \ (\text{mod}2) \\ 3, & \text{if } i \equiv 2 \ (\text{mod}4) \\ 1, & \text{if } i \equiv 0 \ (\text{mod}4) \end{cases}$$

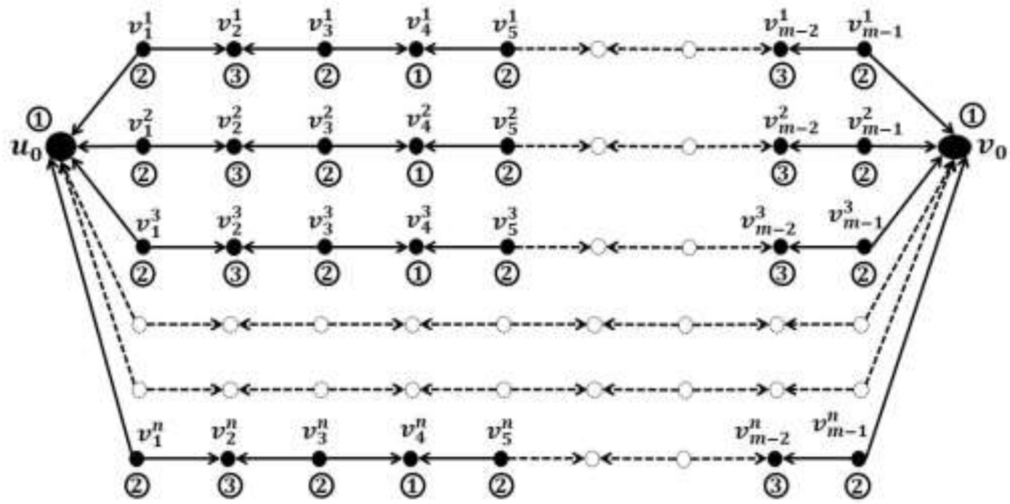

Fig. 2. Star-in-coloring of $\theta(m, n)$, $m \equiv 0 \ (\text{mod}4)$ or $m \equiv 3 \ (\text{mod}4)$

Fom the above two cases, we conclude that the general theta graph $\theta(m, n)$ is star–in–colored and its star–in–chromatic number is $\chi_{si}\left(\theta(m, n)\right) = 3$

Theorem 3. The corona $C_3 \text{ o } P_n$ is star–in–coloring for all odd n except $n = 5,7$.

Proof: Consider the corona graph $C_3 \text{ o } P_n$ with $3(n + 1)$ vertices and $6n$ edges. The vertices of C_3 are denoted by u_1, u_2, and u_3. The vertices of P_n are denoted by v_i^j, where v_i^j represents the i^{th} vertex in the j^{th} copy of P_n attached to u_j, where $1 \leq i \leq n$ and $j = 1,2,3$.

Let V be the vertex set of $C_3 \text{ o } P_n$ and E be the edge set of $C_3 \text{ o } P_n$. We define a function $f: V \rightarrow \{1, 2, 3, \dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon E$, as the following two cases:

Case 1: $n = 1$

$$f(u_j) = j, j = 1,2,3$$

$$f\left(v_i^j\right) = \begin{cases} j + 1, & \text{if } j = 1,2 \\ j - 2, & \text{if } j = 3 \end{cases}$$

Case 2: $n = 3$ or $n \geq 9$

Sub case 2.1: $j = 1$

$$f\left(v_i^j\right) = \begin{cases} 4, & \text{if } i \equiv 1 \ (\text{mod}2) \\ 2, & \text{if } i \equiv 2 \ (\text{mod}4) \\ 5, & \text{if } i \equiv 0 \ (\text{mod}4) \end{cases}$$

Sub case 2.2: $j = 2$

$$f\left(v_i^j\right) = \begin{cases} 4, & \text{if } i \equiv 1 \ (\text{mod}2) \\ 3, & \text{if } i \equiv 2 \ (\text{mod}4) \\ 5, & \text{if } i \equiv 0 \ (\text{mod}4) \end{cases}$$

Sub case 2.3: $j = 3$

$$f\left(v_i^j\right) = \begin{cases} 4, & \text{if } i \equiv 1 \ (\text{mod}2) \\ 1, & \text{if } i \equiv 2 \ (\text{mod}4) \\ 5, & \text{if } i \equiv 0 \ (\text{mod}4) \end{cases}$$

From the above two cases, we conclude that the graph C_3 o P_n is star-in-colored and its star-in-chromatic number satisfies the inequality $3 \leq \chi_{si}$ (C_3 o P_n) ≤ 5.

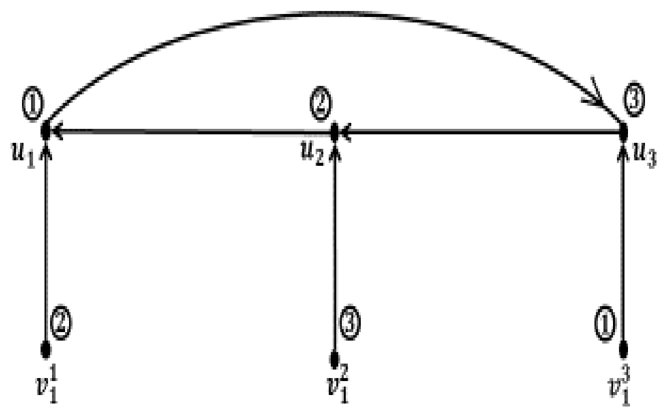

Fig. 3. Star-in-coloring of C_3 o P_1

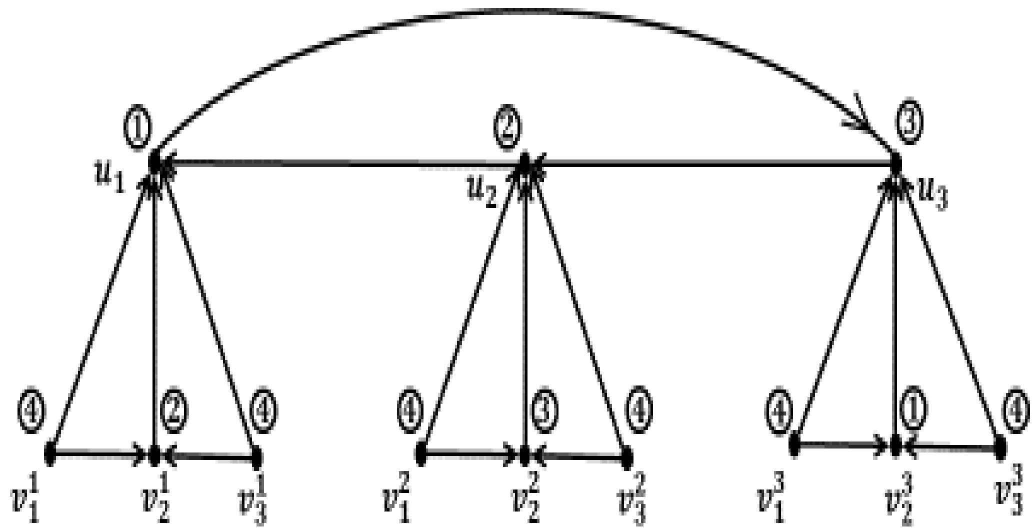

Fig. 4. Star-in-coloring of C_3 o P_3

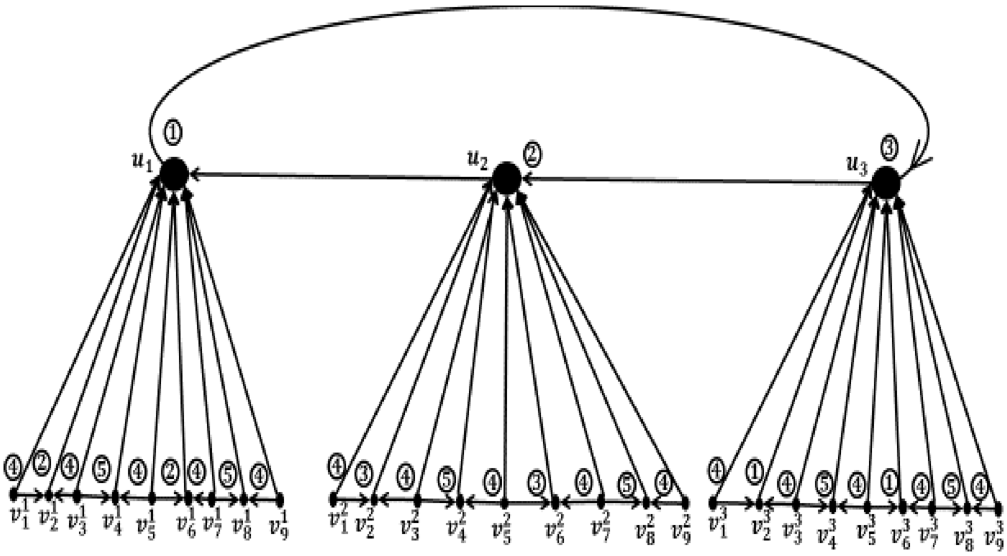

Fig. 5. Star-in-coloring of C_3 o P_9

Theorem 4. The corona C_m o P_n is star–in–coloring for all $m \equiv 0\ (\text{mod}2)$ and for all odd $n \geq 9$.

Proof: Consider the corona graph C_m o P_n with $m(n+1)$ vertices and $2mn$ edges. The vertices of C_m are denoted by $u_1, u_2, u_3, \dots, u_m$ and the vertices of P_n are denoted by v_i^j, where v_i^j represents the i^{th} vertex in the j^{th} copy of P_n attached to u_j, where $1 \leq i \leq n$ and $1 \leq j \leq m$.

Let V be the vertex set of C_m o P_n and E be the edge set of C_m o P_n. We define a function $f: V \to \{1, 2, 3, \dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \in E$, as the following two cases:

Case 1: $m \equiv 0\ (\text{mod}4)$

$$f(u_j) = \begin{cases} 1, & \text{if } j \equiv 1\ (\text{mod}2) \\ 2, & \text{if } j \equiv 2\ (\text{mod}4) \\ 3, & \text{if } j \equiv 0\ (\text{mod}4) \end{cases}$$

Sub case 1.1: $j \equiv 1(\text{mod}2)$

$$f\left(v_i^j\right) = \begin{cases} 4, & \text{if } i \equiv 1\ (\text{mod}2) \\ 5, & \text{if } i \equiv 2\ (\text{mod}4) \\ 6, & \text{if } i \equiv 0\ (\text{mod}4) \end{cases}$$

Sub case 1.2: $j \equiv 2\ (\text{mod}4)$

$$f\left(v_i^j\right) = \begin{cases} 3, & \text{if } i \equiv 1\ (\text{mod}2) \\ 4, & \text{if } i \equiv 2\ (\text{mod}4) \\ 5, & \text{if } i \equiv 0\ (\text{mod}4) \end{cases}$$

Sub case 1.3: $j \equiv 0(\text{mod}4)$

$$f\left(v_i^j\right) = \begin{cases} 2, & \text{if } i \equiv 1\ (\text{mod}2) \\ 4, & \text{if } i \equiv 2\ (\text{mod}4) \\ 5, & \text{if } i \equiv 0\ (\text{mod}4) \end{cases}$$

Case 2: $m \equiv 2\ (\text{mod}4)$

$$f(u_j) = \begin{cases} 1, & \text{if } j \equiv 1 \ (\text{mod}2) \\ 2, & \text{if } j \equiv 2 \ (\text{mod}4) \\ 3, & \text{if } j \equiv 3 \ (\text{mod}4) \\ 4, & \text{if } j = m \end{cases} \text{ and } j \neq m$$

Sub case 2.1: $j \equiv 1,2 \ (\text{mod}6)$

$$f\left(v_i^j\right) = \begin{cases} 3, & \text{if } i \equiv 1 \ (\text{mod}2) \\ 5, & \text{if } i \equiv 2 \ (\text{mod}4) \\ 6, & \text{if } i \equiv 0 \ (\text{mod}4) \end{cases}$$

Sub case 2.2: $j \equiv 3,4 \ (\text{mod}6)$

$$f\left(v_i^j\right) = \begin{cases} 4, & \text{if } i \equiv 1 \ (\text{mod}2) \\ 5, & \text{if } i \equiv 2 \ (\text{mod}4) \\ 6, & \text{if } i \equiv 0 \ (\text{mod}4) \end{cases}$$

Sub case 2.3: $j \equiv 0,5 \ (\text{mod}6)$

$$f\left(v_i^j\right) = \begin{cases} 2, & \text{if } i \equiv 1 \ (\text{mod}2) \\ 5, & \text{if } i \equiv 2 \ (\text{mod}4) \\ 6, & \text{if } i \equiv 0 \ (\text{mod}4) \end{cases}$$

From the above two cases, we conclude that the corona graph $C_m \text{ o } P_n$ is star-in-colored and its star-in-chromatic number is $\chi_{si} (C_m \text{ o } P_n) = 6$.

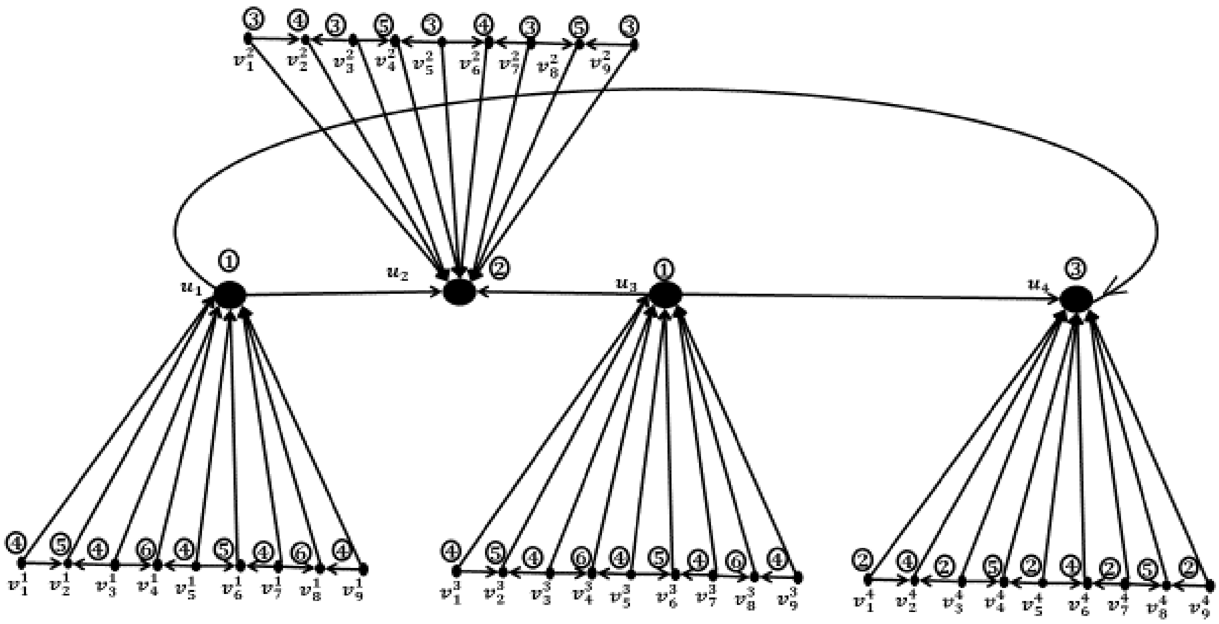

Fig. 6. Star-in-coloring of C_4 o P_9

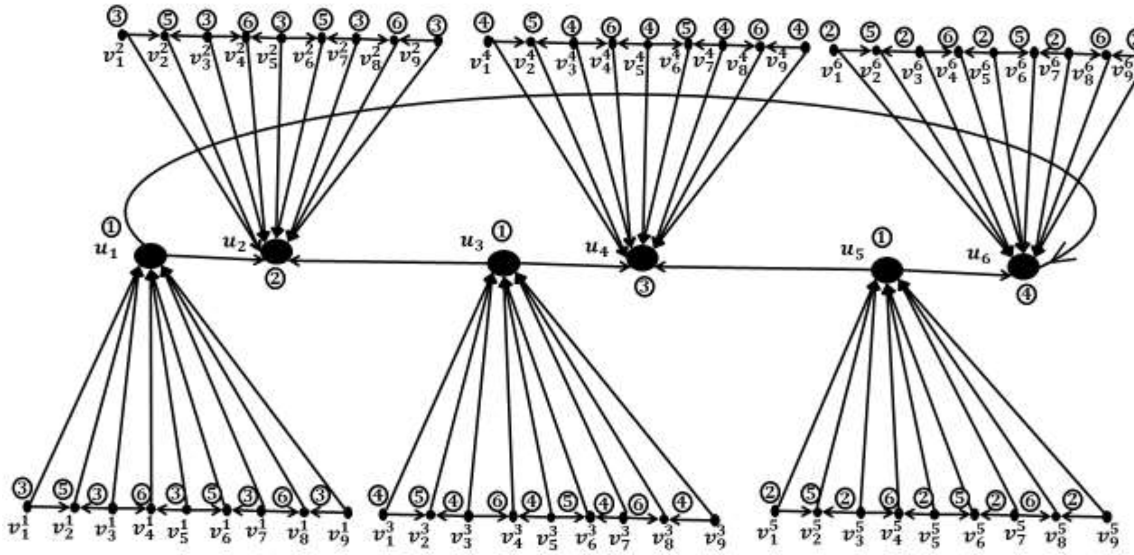

Fig. 7. Star-in-coloring of C_6 o P_9

3. CONCLUSION

The findings of this study are summarized in the following results:

1. χ_{si} $(H_s) = 5$.
2. χ_{si} $(\theta(m,n)) = 3$
3. $3 \le \chi_{si}$ $(C_3$ o $P_n) \le 5$, for all odd n except $n = 5,7$
4. χ_{si} $(C_m$ o $P_n) = 6$, for all $m \equiv 0$ (mod2) and for all odd $n \ge 9$

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