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Dynamical Modeling of Forest Depletion under Population and Socio-Economic Pressures for Environmental Sustainability

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Abstract
Forest resources, which are essential for biodiversity conservation, climate regulation and ecosystem stability facing severe threats from human activities like population increase, industrialization and socioeconomic pressures. This research work presents a novel nonlinear dynamical model consisting of six coupled differential equations for analyzing the depletion and conservation of forest resources. The model takes into account several control variables including forest biomass, human population, industrial development, reforestation efforts, forest fires and socio-economic pressure. Stability analysis theory is used to analyze the dynamical system of the model. The ODE45 function in MATLAB is used to perform the numerical solutions and simulation of the system. The results are displayed using graphs and biologically interpretated. The depletion of forest resources is observed as result of growing population density and related pressure. However, by limiting socio-economic pressure, man-made fire, significant plantation, optimal industrial development and utilizing technology, forest can be preserved.

Keywords: *Dynamical model, Equilibrium point, Forest resources, numerical simulation, Population growth, Stability analysis.*

INTRODUCTION

Forests also known as the 'lungs of the earth' and provide important ecosystem services like biodiversity conservation, climate regulation, carbon sequestration and raw material for industrial development. They are not only the essential component of ecological stability but also a vital resource for human being survival. Despite their immense value, forest resources are under increasing pressure from human activities such as deforestation, industrialization, forest fires and unsustainable land use practices. The degradation of forests is one of the most serious environmental issues of the twenty first century. In India, which has approximately 69 million hectare of forest cover are under enormous pressure from population growth and its associated pressure. 833 million people or nearly 69% of the population live in rural areas and depends on forests for their livelihood. According to (Hussen Teru & Rao Koya, 2020), forest ecosystems are increasingly under threats due to human population growth, industry development and natural disasters like forest fires, which have become more frequent and intense. The current rate of environmental change with anthropogenic pressure has altered the earth's ability to support sustainable forest ecosystems (Matia et al., 2023). This issue transcends national boundaries and requires global attention on forest cover loss, which has a global impact on climate regulation and biodiversity at a planetary scale. While industrial development has contributed to economic growth, it also had a negative impact on forest resources depletion. Mining, logging and urbanization have accelerated deforestation which resulting in habitat destruction and increased greenhouse gas emission (Misra et al., 2014). Furthermore, illegal logging, agriculture land expansion and land conversion promote the depletion of forest resources. Without intervention, these trends have the potential to cause irreversible damage to ecosystem which affecting millions of people's livelihoods and undermining climate change efforts(Gibson et al., 2000; Lata & Misra, 2017).

Numerous studies have explored the complex interaction between human activity and forest ecosystem for sustainable management techniques. For example, (Shukla & Dubey, 1997) proposed a model that integrate the effect of population growth, pollution and forest resources depletion. They highlighted the negative effect of unchecked population growth and pollution. (Misra et al., 2014) explored the interplay between population growth, industrialization and forest biomass through dynamical model that emphasizing the importance of sustainable policies to maintain ecological balance. Similarly, the effect of population pressure, industrial activities on forest resources through mathematical models was investigated by (Didiharyono & Kasse, 2021; Eswari et al., 2019; Ezeorah & Ekaka-A, 2022; Misra et al., 2014; Ramdhani et al., 2015; Shukla & Misra, n.d.). The stability analysis revealed that increased industrial activities and crowding led to lower forest density and underscoring the detrimental effects of unregulated industrial expansion. (Sebastian & Victor, 2017, 2018) emphasize that the increasing demand for agriculture land, timber and other forest resources by growing population has led to significant forest degradation. In India rural population largely depends on forests resources for their living which put additional stress on these ecosystems (Pratama et al., 2020). They developed a two-patch model to capture the effect of population pressure on forest ecosystem and revealing the significant regional variation in resources stress. (Aquino & Bologna, 2021; Dipesh et al., 2023) underscores the quantitative relationship between industrial development and forest cover loss and pointed to the need for stringent regulations to balance economic growth and environmental sustainability. (Goshu & Endalew, 2022) introduced a model that incorporates reforestation efforts as a control variable and demonstrated how active conservation measure can stabilize forest resources. Similarly, (Altamirano-Fernández et al., 2023; Dipesh et al., 2023; Ezeorah & Ekaka-A, 2022) employed mathematical models to optimize industrial growth through sustainable forest biomass harvesting. These studies emphasize the importance of balancing human activities with ecological preservation. (Fanuel et al., 2024; Fanuel, Mirau, Kajunguri, et al., 2023; Fanuel, Mirau, Mayengo, et al., 2023) employed mathematical model to account for the uncertainty in environmental data and providing a flexible approach to predicting forest ecosystem dynamics. By integrating different ecological factors these models provide valuable insight into sustainable management strategies.As far as we aware, the combined effect of population growth, industrial development, reforestation, forest fires and socio-economic pressure on the conservation of forest biomass have been not investigated. Due to this, we proposed and analyzed a dynamical model to study impact of population growth and socio-economic pressure for the conservation of depleted forestry resources.

MATHEMATICAL MODEL DESCRIPTION AND FORMULATION

To model the depletion of forestry resources with a system of six nonlinear differential equations, we will be considering factors affecting forest growth such as population growth, population pressure, forest fires, industrial development, plantation activities, and socio-economic factors. The model is constructed with the following compartments:

- **Forest Area (F)**: Represents the forest cover or biomass.
- **Human Population (P)**: Represents the human population affecting forest dynamics.
- **Industrial Development (I)**: Represents industrial activities that degrade forest areas.
- **Reforestation and Plantation (R)**: Represents efforts to replant and conserve forest areas.
- **Forest Fires (A)**: Represents the impact of natural or human-induced forest fires.
- **Socio-economic Pressure (S)**: Represents socio-economic activities that exert pressure on forest conservation, such as agriculture, illegal logging, and land conversion.

Furthermore, the model can be formulated with consideration of the following assumptions-

- Human population and population pressure are considered, and Population pressure rate is directly related to population density.
- The forest area **F** grows naturally at a rate r and is affected negatively by population growth, industrial development, forest fires, and socio-economic pressures.
- The human population **P** grows at a rate α and impacts forest cover through consumption, land conversion, and increased pressure.
- Industrial development **I** is fuelled by population growth and affects forest cover by land conversion, pollution, and resource extraction.
- Reforestation **R** contributes positively to forest cover through conservation efforts, plantation activities, and restoration projects.
- Forest fires **A** increase due to socio-economic factors, industrial activities, and climate change, negatively impacting forest cover.
- Socio-economic pressure **S** increases due to population growth, industrial development, and inadequate conservation measures.

Using above assumption, the model equations with initial conditions is given by

$$\left.\begin{aligned}
\frac{dF}{dt} &= rF - \beta\frac{FP}{N} - \gamma IF - \delta AF - \zeta SF + \eta FR, && F(0)=F_0 \geq 0\\
\frac{dP}{dt} &= \alpha P\left[1-\frac{P}{K}\right] - \phi\frac{FP}{N}, && P(0)=P_0 \geq 0\\
\frac{dI}{dt} &= \sigma P - \tau IF - \mu I, && I(0)=I_0 \geq 0\\
\frac{dR}{dt} &= \rho\frac{FP}{N} - \lambda R, && R(0)=R_0 \geq 0\\
\frac{dA}{at} &= \theta S + \xi I - \kappa A, && A(0)=A_0 \geq 0\\
\frac{dS}{dt} &= \psi P + \omega I - \nu S, && S(0)=S_0 \geq 0
\end{aligned}\right\} \quad (1)$$

In model system (1), $F(t)$ is the density of forest biomass with natural growth rate r, $P(t)$ is density of human population with growth rate α and carrying capacity *K*, $I(t)$ represent industrial development, $R(t)$ represents efforts for reforestation and plantation, $A(t)$ is the impact of forest fires, *S(t)* represents socio-economic pressure and activities. β is the Rate at which population pressure reduces forest cover, γ is rate of forest degradation due to industrial development, δ is the rate of forest loss due to forest fires and η represents Rate of reforestation and plantation efforts. The Effect of forest availability on population migration or reduction is denoted by ϕ, σ represents Rate of industrial growth due to population, τ is the Rate of industrial mitigation through sustainable practices or regulations, μ is the Natural decay or regulation of industrial activities, ρ is the Rate of reforestation activities driven by population pressure, λ is the Rate at which reforestation efforts decline or fail, θ is the Rate of forest driven by socio economic factors, ξ is the Rate of fire increase due to industrial activities and κ represents Natural decay or control measure of forest fires. Furthermore, ψ represent Rate of socio-economic pressure increase due to population, ω is the Rate of socio-economic pressure increase due to industrial activities, ν is the Mitigation rate of socio-economic pressure through policies or sustainable practices and N represents total relevant population affecting forest dynamics.

POSITIVITY OF THE MODEL'S SOLUTION

Theorem: If $F(0)>0,\ P(0)>0,\ I(0)>0,\ R(0)>0,\ A(0)>0,\ S(0)>0$ in the feasible set Δ, then the solution set $\{F(t),P(t),I(t),R(t),A(t),S(t)\}$ of model (1) is positive for all $t\geq 0$.

Proof: From the first equation of model (1), it can write in inequality from $\frac{dF}{dt}=rF-\beta\frac{FP}{N}-\gamma IF-\delta AF-\zeta SF+\eta FR\geq-\beta\frac{FP}{N}-\gamma IF-\delta AF-\zeta SF$. After some simplification, we get analytic solution $F(\mathrm{t})>F(0)\mathrm{e}^{rt}>0$. Since $F(0)$ is constant of integrations and the represent initial value of forest area and a positive quantity. Since analytical solution leads to $t\to\infty$ and $F(t)>0$. By simplify we can conclude that forest area

$F(t)$ is always positive. From the second equation of model (1), it can write in inequality from $\frac{dP}{dt}=\alpha P\left[1-\frac{P}{K}\right]-\phi\frac{FP}{N}\geq-\alpha\frac{P^2}{K}-\phi\frac{FP}{N}$ After some simplification, we get analytic solution $P(\mathrm{t})>P(0)e^{\alpha t}$. Since $P(0)$ is constant of integrations and the represent initial value of human population and a positive quantity. Since analytical solutions leads to $t\to\infty$ and $P(t)>0$. By simplify the equation we can conclude that human population $P(t)$ is always positive. From the third equation of model (1), it can write inequality from the equation $\frac{dI}{dt}=\sigma P-\tau IF-\mu I\geq-\tau IF-\mu I$. After some simplification, we get analytic solution $I(t)>I(0)e^{(\tau-\mu)t}$. Since $I(0)$ is constant of integrations and represent initial value of industrial development is a positive quantity. Analytic solution leads to $t\to\infty$ and $I(t)>0$. By simplify the equation, we conclude industrial development $I(t)$ is always positive. From the fourth equation of the model (1), it can write inequality form the equation $\frac{dR}{dt}=\rho\frac{FP}{N}-\lambda R\geq-\lambda R$. After some simplification and integrating the equation, we get analytic solution $R(\mathrm{t})>\mathrm{R}(0)\mathrm{e}^{\lambda t}$. Since $R(0)$ is constant of integrations and represent initial value of Reforestation and plantation is positive quantity. Analytic solution leads to $t\to\infty$ and $R(t)>0$. By simplify the equation, we conclude reforestation $R(t)$ is always positive. From the fifth equation of the model (1), it can write inequality from the equation $\frac{dA}{dt}=\theta S+\xi I-\kappa A\geq-\kappa A$. After some simplification and integrating the equation, we get analytic solution $A(t)>A(0)e^{\kappa t}$. Since $A(0)$ is constant of integrations and represent initial value of Forest fires $A(t)$ is positive quantity. Analytic solution leads to $t\to\infty$ and $A(t)>0$. By simplify the equation, we conclude forest fires $A(t)$ is always positive. Finally, sixth equation of the model (1), it can write inequality from the equation $\frac{dS}{dt}=\psi P+\omega I-vS\geq-vS$. After some simplification and integrating the equation, we get analytic solution $\mathrm{S}(t)>S(0)e^{vt}$. Since $S(0)$ is constant of integrations and represent value of socio-economic pressure $S(t)$ is always positive quantity. Analytic solution leads to $t\to\infty$ and $S(t)>0$. By simplify the equation, we conclude socio-economic pressure $S(t)$ is always positive. Therefore, the solution set $\{F(t), P(t), I(t), R(t), A(t), S(t)\}$ of system (1) is all nonnegative for all $t \geq 0$. Hence in biological validity of the model, if the population such as density of Forest area, density of Human population, Industrialization, Reforestation and Plantation, Forest fires and Socio-economic pressure are negative. Then they are not biologically feasible, so that checked the positivity of system. Hence the theorem is proved.

BOUNDEDNESS OF SOLUTION OF THE MODEL

The area of attraction is defined as the measurement of defined set distance from the mechanical phenomena which can start and point of stability and convergence by using the Lyapunov's function.

Lemma 1. The set

$V = \{(F,P,I,R,A,S): 0 \leq F \leq M, 0 \leq P \leq K, 0 \leq I \leq \pi, 0 \leq R \leq \varepsilon, 0 \leq A \leq A_m, 0 \leq S \leq S_n\}$ is a region of attraction for all solution initiating in the interior of the positive octant.

Since, $\pi = \dfrac{\sigma K}{\tau F + \mu}$, $\varepsilon = \rho \dfrac{MK}{N\lambda}$, $A_m = \dfrac{\theta S + \xi I}{\kappa}$, $S_n = \dfrac{\psi K + \omega I}{\nu}$, $M = \dfrac{\eta R}{r}$

Proof. From the first equation of system (1) $\dfrac{dF}{dt} = rF - \beta \dfrac{FP}{N} - \gamma IF - \delta AF - \zeta SF + \eta FR$. Since $F(\text{t})$ is positive which implies that $\dfrac{dF}{dt} \geq 0$ or $0 \leq \dfrac{dF}{dt}$. By comparison theorem, $0 \leq \dfrac{dF}{dt} \leq rF - \beta \dfrac{FP}{N} - \gamma IF - \delta AF - \zeta SF + \eta FR$, which implies that $0 \leq rF - \beta \dfrac{FP}{N} - \gamma IF - \delta AF - \zeta SF + \eta FR$. After some simplification we get, $F \leq \dfrac{\eta R}{r}$. Depending on the definition $0 \leq F_0 \leq F(t)$ and $0 \leq F(t)$. Which implies that, $0 \leq F(t) \leq \dfrac{\eta R}{r}$. Let $M = \dfrac{\eta R}{r}$ then $0 \leq F_0 \leq F(t)$ implies $F(\text{t}) \leq \dfrac{\eta R}{r}$. From the second equation of the system (1) $\dfrac{dP}{dt} = \alpha P\left[1 - \dfrac{P}{K}\right] - \phi \dfrac{FP}{N}$. Since $P(\text{t})$ is positive which implies that $\dfrac{dP}{dt} \geq 0$ or $0 \leq \dfrac{dP}{dt}$ by comparison theorem $0 \leq \alpha P\left[1 - \dfrac{P}{K}\right] - \phi \dfrac{FP}{N}$. Now $0 \leq \alpha P\left[1 - \dfrac{P}{K}\right]$. After simplification we get $P \leq K$. Now we have $0 \leq P(\text{t}) \leq K$ implies that $P(\text{t}) \leq K$. From the third equation of the system (1) $\dfrac{dI}{dt} = \sigma P - \tau IF - \mu I$ now $I(\text{t})$ is positive and implies that $\dfrac{dI}{dt} \geq 0$ or $0 \leq \dfrac{dI}{dt}$. Now $0 \leq \sigma P - \tau IF - \mu I$ after some simplification, $0 \leq I \leq \pi$ since $\pi = \dfrac{\sigma K}{\tau F + \mu}$. From the fourth equation of system (1) $\dfrac{dR}{dt} = \rho \dfrac{FP}{N} - \lambda R$, since $R(\text{t})$ is positive which implies that $\dfrac{dR}{dt} \geq 0$ or $0 \leq \dfrac{dR}{dt} \leq \rho \dfrac{M \in}{N} - \lambda R,$ after some simplification $0 \leq R(\text{t}) \leq \varepsilon$, since $\varepsilon = \rho \dfrac{MK}{N\lambda}$. Fifth equation of system (1) $\frac{dA}{dt}=\theta S+\xi I-\kappa A$. Since $A(\mathrm{t})$ is positive and implies that $\frac{dA}{dt}\geq 0$ and $0\leq\frac{dA}{dt}$. Now $0\leq\theta S+\xi I-\kappa A$, this show that $0\leq A(\mathrm{t})\leq \mathrm{A}_m$, where $A_m=\frac{\theta S+\xi I}{\kappa}$. From sixth equation of the system (1) $\frac{dS}{dt}=\psi P+\omega I-vS$, Since $S(\mathrm{t})$ is positive and implies that $\frac{dS}{dt}\geq 0$ and $0\leq\frac{dS}{dt}\leq\psi P+\omega I-vS$ which means $0\leq S(t)\leq S_n$ where, $S_n=\frac{\psi K+\omega I}{v}$. This indicate that a region of attraction for all solution initiation in the interior of positive octant. Hence it is bounded. The V shows that all solutions of model are nonnegative and bounded, and model is biologically well behaved.

5. STABILITY ANALYSIS OF EQUILIBRIUM POINTS OF THE MODEL

The Jacobian matrix of the model system (1) is given by

$$J=\begin{bmatrix} r-\beta\frac{P}{N}-\gamma I-\delta A-\zeta S+\eta R & -\beta\frac{F}{N} & -\gamma F & \eta F & -\delta F & -\zeta F \\ -\phi\frac{P}{N} & \alpha\left(1-\frac{2P}{K}\right)-\phi\frac{F}{N} & 0 & 0 & 0 & 0 \\ -\tau I & \sigma & -\tau F-\mu & 0 & 0 & 0 \\ \rho\frac{P}{N} & \rho\frac{F}{N} & 0 & -\lambda & 0 & 0 \\ 0 & 0 & \xi & 0 & -\kappa & 0 \\ 0 & \varphi & \omega & 0 & 0 & -v \end{bmatrix}$$

Let considered a nonlinear differential equation of model (1) to determine equilibrium point with $F=0, P=0$ and $I=0$, into all equation of model (1), we have $F=0, P=0, I=0, R=0, A=0, S=0$. Then the equilibrium point is $E_0\left(0,0,0,0,0,0\right)$. If the value $P=0$, then the equilibrium points $E_1\left(0,0,0,0,0,0\right)$. If $F=0$, Then we have two cases, firstly is equilibrium points $E_2\left(0,0,0,0,0,0\right)$, second equilibrium points $E_2'\left(0,\mathrm{P}_2,\mathrm{I}_2,0,\mathrm{A}_2,S_2\right)$. The equilibrium points $E_0\left(0,0,0,0,0,0\right)$, $E_1\left(0,0,0,0,0,0\right)$ and $E_2\left(0,0,0,0,0,0\right)$ show trivial equilibrium points that means a small perturbation will cause the system move away and the system is unstable. For the equilibrium point $E_2'\left(0,\mathrm{P}_2,\mathrm{I}_2,0,\mathrm{A}_2,S_2\right)$ where $P_2=K, I_2=\frac{\sigma K}{\mu}, A_2=\frac{\theta S+\xi I}{\kappa}, S_2=\frac{\varphi\kappa+\omega I}{v}$, the characteristic equation of the determinant of the Jacobean matrix J for model (1) at $E_2\left(0,\mathrm{P}_2,\mathrm{I}_2,0,\mathrm{A}_2,S_2\right)$ is $(c-a_2)(c-b_2)\left(c+\tau F+\mu\right)\left(c+\lambda\right)\left(c+k\right)\left(c+v\right)=0$ where $b_2=\left(\alpha-\phi\frac{F}{N}-\frac{2P_2}{K}\right)$ and

$c = a_2,\ b_2 > 0,\ \mu > \tau F,\ \alpha,\ \phi\dfrac{F}{N}$ are the eigenvalues in which two eigenvalues are positive , that mean $E_2(0, P_2, I_2, R_2, A_2, S_2)$ become unstable.

Now assuming that $F \neq 0, P \neq 0, I \neq 0, R \neq 0, A \neq 0, S \neq 0$. Then $E_3(F_3, P_3, I_3, R_3, A_3, S_3)$ is interior equilibrium when all variables exist. Since, $F_3 = \dfrac{r}{\zeta S + \beta\dfrac{P}{N} + \gamma I + \delta A + \eta R}, P_3 = K\left(1 - \phi\dfrac{F}{\alpha N}\right), I_3 = \dfrac{\sigma P}{\tau F + \mu}, R_3 = \rho\dfrac{FP}{\lambda N},$
$A_3 = \dfrac{\theta S + \xi I}{\kappa}$, and $S_3 = \dfrac{\varphi P + \omega I}{\nu}$

The equilibrium point $E_3(F_3, P_3, I_3, R_3, A_3, S_3)$ is nontrivial and difficult to analyze using the Jacobean matrix. To analysis the local and global asymptotically stability, we use the following theorems with different assumption.

Theorem 1: The interior equilibrium $E_3\left(F_3, P_3, I_3, R_3, A_3, S_3\right)$ if exits, is locally asymptotically stable in region V.

Proof. Nontrivial equilibrium E_3 is difficult to determine the nature of eigen values of Jacobin matrix, by using Routh-Hurwitz criterion, so we use Lyapunov stability theory. By using the Taylors series expansion, the linearized system of model (1) at the equilibrium point $E_3\left(F_3, P_3, I_3, R_3, A_3, S_3\right)$ can be transformed as $F = F_3 + f_1, \mathrm{P} = P_3 + \mathrm{p}_1, \mathrm{I} = \mathrm{I}_3 + i_1, R = R_3 + r_1, A = A_3 + a_1, S = S_3 + s_1$

Where $f_1, p_1, i_1, r_1, a_1, s_1$ are perturbations. By applying the positive definite function, we have

$$W(F, P, I, R, A, S) = \frac{1}{2}\left(F^2 + P^2 + I^2 + R^2 + A^2 + S^2\right)$$

$$W = \frac{1}{2}\left(\frac{n_1 f_1^2}{F_3} + \frac{n_2 p_1^2}{P_3} + n_3 i_1^2 + n_4 r_1^2 + n_5 a_1^2 + n_6 s_1^2\right) \tag{2}$$

$$\frac{dW}{dt} = \left(\frac{n_1 f_1 \bar{F}}{F_3} + \frac{n_2 p_1 \bar{P}}{P_3} + n_3 i_1 \bar{I} + n_4 r_1 \bar{R} + n_5 a_1 \bar{A} + n_6 s_1 \bar{S}\right) \tag{3}$$

Substituting the values of $\dfrac{dF}{dt}, \dfrac{dP}{dt}, \dfrac{dI}{dt}, \dfrac{dR}{dt}, \dfrac{dA}{dt}, \dfrac{dS}{dt}$ into this equation,

$$\frac{dW}{dt} = n_1 f_1\left(r - \beta\frac{P}{N} - \gamma I - \delta A - \zeta S + \eta R\right) + n_2 p_2\left(\alpha - \frac{\alpha P}{K} - \phi\frac{F}{N}\right) + n_3 i_1\left(\sigma P - \tau F - \mu\right)$$
$$n_4 r_1\left(\rho\frac{FP}{N} - \lambda\right) + n_5 a_1\left(\theta S + \xi I - \kappa\right) + n_6 s_1\left(\varphi P + \omega I - \nu S\right)$$

$$\Rightarrow \; -n_1\beta\frac{P}{N}f_1^2-n_2\frac{\alpha}{K}p_1^{\;2}-n_3\left(\tau F+\mu\right)i^2{}_1-n_4\lambda r_1^2-n_5\kappa a_1^{\;2}-n_6 v s_1^{\;2}$$
$$-\left(n_1\beta-n_4\rho\right)f_1r_1-\left(n_2\phi-n_4\rho\right)p_1r_1-n_1\left(\delta a_1f_1+\eta r_1f_1-ri_1f_1-\zeta f_1s_1\right) \tag{4}$$
$$+n_3\tau i_1f_1-n_3\sigma p_1i_1-n_5\xi a_1i_1-n_6\omega i_1s_1$$

Now choosing $n_1=1,n_2=\dfrac{\beta_1}{\beta},n_3=1,n_4=1,n_5=\dfrac{\phi_1}{\phi},n_6=1$

$$\Rightarrow -\beta\frac{P}{N}f_1^2-\frac{\beta_1}{\beta}\frac{\alpha}{K}p_1^{\;2}-\left(\tau F+\mu\right)i_1^{\;2}-\lambda r_1^{\;2}-\frac{\phi_1}{\phi}\kappa a_1^{\;2}-nvs_1^{\;2}$$
$$-\left(\beta-\rho\right)f_1r_1-\left[\left(\frac{\beta_1}{\beta}\right)\phi-\rho\right]p_1r_1-\left[\delta a_1f_1+\eta r_1f_1+\zeta s_1f_1-\left(r+\tau\right)f_1i_1\right] \tag{5}$$
$$+\sigma p_1i_1-\frac{\phi_1}{\phi}\xi a_1i_1-\omega i_1s_1$$

$\dfrac{dW}{dt}<0$ will be negative definite inside the region of attraction V. this implies that $E_3\left(F_3,P_3,I_3,R_3,A_3,S_3\right)$ is local asymptotically stable.

Theorem 2: The interior equilibrium $E_3\left(F_3,P_3,I_3,R_3,A_3,S_3\right)$ if exits, is Globally asymptotically stable in region V.

Proof. Let us apply Lyapunov function in system (1)

$$W=\left[F-F_3-F_3In\frac{F}{F_3}\right]+a\left[P-P_3-P_3In\frac{P}{P_3}\right]+\frac{b\left(I-I_3\right)^2}{2}+\frac{c\left(R-R_3\right)^2}{2}+\frac{d\left(A-A_3\right)^2}{2}+\frac{e\left(S-S_3\right)^2}{2} \tag{6}$$

where a,b,c,d are positive constant with an approximate value. This equation is a positive definite function if

$$W=\left[F-F_3-F_3In\frac{F}{F_3}\right]+a\left[P-P_3-P_3In\frac{P}{P_3}\right]+\frac{b\left(I-I_3\right)^2}{2}+\frac{c\left(R-R_3\right)^2}{2}+\frac{d\left(A-A_3\right)^2}{2}+\frac{e\left(S-S_3\right)^2}{2}>0. \tag{7}$$

Differentiate Equation (1) with respect to time t, we get

$$\frac{dW}{dt}=\left(F-F_3\right)\frac{\dot{F}}{F}+c_1\left(P-P_3\right)\frac{\dot{P}}{P}+c_2\left(I-I_3\right)\bar{I}+c_3(R-R_3)\bar{R}+c_4(A-A_3)\bar{A}+c_5(S-S_3)\bar{S} \tag{8}$$

Now using the system of differential equations and applying Jacobean matrix, we get

$$\Rightarrow -\beta\frac{P}{N}\left(F-F_3\right)^2 - a\left(\frac{\alpha}{K}\right)\left(P-P_3\right)^2 - b\left(\tau F+\mu\right)\left(I-I_3\right)^2 - c\lambda(R-R_3)^2 - d\kappa(A-A_3)^2 - ev(\mathrm{S}-S_3)^2$$
$$-\left(\beta-a\rho\right)\left(F-F_3\right)(R-R_3)-\left(\phi-b\rho\right)\left(P-P_3\right)(R-R_3)-rc\left(I-I_3\right)\left(F-F_3\right)-\delta\left(A-A_3\right)\left(F-F_3\right)$$
$$-\zeta\left(A-A_3\right)\left(F-F_3\right)+\eta\left(R-R_3\right)\left(F-F_3\right)+\tau\left(I-I_3\right)\left(F-F_3\right)-\sigma d\left(P-P_3\right)\left(I-I_3\right)$$
$$-\xi\left(A-A_3\right)\left(I-I_3\right)-\omega e\left(I-I_3\right)\left(S-S_3\right)$$

Then, let us consider, $a=\frac{\beta_1}{\beta}, b=c=1, d=\frac{\phi_1}{\phi}, e=1$ and substituting into equation (8)

We get

$$\frac{dW}{dt} = -\beta\frac{P}{N}\left(F-F_3\right)^2 - \frac{\beta_1}{\beta}\left(\frac{\alpha}{K}\right)\left(P-P_3\right)^2 - \left(\tau F+\mu\right)\left(I-I_3\right)^2 - \lambda(R-R_3)^2 - \frac{\phi_1}{\phi}\kappa(A-A_3)^2 - v(\mathrm{S}-S_3)^2$$
$$-\left(\beta-\left(\frac{\beta_1}{\beta}\right)\rho\right)\left(F-F_3\right)(R-R_3)-\left(\phi-\rho\right)\left(P-P_3\right)(R-R_3)-r\left(I-I_3\right)\left(F-F_3\right)-\delta\left(A-A_3\right)\left(F-F_3\right)$$
$$-\zeta\left(S-S_3\right)\left(F-F_3\right)+\eta\left(R-R_3\right)\left(F-F_3\right)+\tau\left(I-I_3\right)\left(F-F_3\right)-\sigma\left(\frac{\phi_1}{\phi}\right)\left(P-P_3\right)\left(I-I_3\right)$$
$$-\xi\left(A-A_3\right)\left(I-I_3\right)-\omega\left(I-I_3\right)\left(S-S_3\right)<0$$

(9)

Which implies $\frac{dW}{dt}<0$. Now, $\frac{dW}{dt}<0$ is negative definite the region of attraction W. This show that the equilibrium point $E_3\left(F_3,P_3,I_3,R_3,A_3,S_3\right)$ is globally asymptotically stable. This theorem implies that under certain condition, the system (1) highly resistance to change in species composition and all specie coexist in same habitat. This indicates that the density of forest resources decreases due to increase of density of population and associate pressure. Generally, the analytical analysis shows that the density of forest resource is extinction in the absence of conservation. The presence of conservation, the density of forest resources are extinction in absence of conservation. The presence of conservation, the density of forest resources can be maintained to its original level.

Numerical Simulation

This part showcases simulations of model (1) performed with MATLAB to assess the practicality of the interior equilibrium and the conditions for its stability. The subsequent parameter values are employed in the system:

$$r=0.6,\beta=0.004,\gamma=0.02,\delta=0.007,\zeta=0.1,\eta=0.01,\alpha=0.8,K=100,\phi=0.05,\sigma=0.04,$$
$$\tau=0.04,\mu=0.08,\rho=0.02,\lambda=0.003,\theta=0.1,\xi=0.003,\kappa=0.04,\psi=0.05,\omega=0.02,v=0.03$$

For these numerical values the interior equilibrium point $E_3\left(F_3,P_3,I_3,R_3,A_3,S_3\right)$ having values $F_3=0.0264, P_3=99.9736, I_3=0.02499, R_3=0.0352, A_3=100.1874, S_3=166.7893$ and corresponding Jacobian matrix J is given by

$$J = \begin{bmatrix} -23.1414 & -0.0000002112 & -0.00528 & 0.000264 & -0.001848 & -0.00264 \\ -0.09997 & -0.1 & 0 & 0 & 0 & 0 \\ -0.009996 & 0.02 & -8.0011 & 0 & 0 & 0 \\ 0.03999 & 0.00001056 & 0 & -0.03 & 0 & 0 \\ 0 & 0 & 0.03 & 0 & -0.04 & 0 \\ 0 & 0.05 & 0.02 & 0 & 0 & -0.03 \end{bmatrix}$$

After calculation of characteristic equation of above matrix, we have eigen values

$$\lambda_1 = -23.1414, \lambda_2 = -8.0011, \lambda_3 = -0.300003, \lambda_4 = -0.0999971, \lambda_5 = -0.04, \lambda_6 = -0.0299995$$

Since all the eigen values are negative Thus, the interior equilibrium point $E_3\left(F_3, P_3, I_3, R_3, A_3, S_3\right)$ is asymptotically stable. The simulation of the proposed forest conservation model shows that all eigenvalues related to the interior equilibrium point possess negative real parts. This suggests that the interior equilibrium is asymptotically stable. In other words, minor disturbances or deviations from this equilibrium will decrease over time, allowing the system to return to its steady state. This finding indicates that, given the specified parameter conditions, the model forecasts a stable coexistence of forest resources alongside influencing factors like population pressure, industrial development and socio-economic pressures, thus endorsing the viability of sustainable forest management.

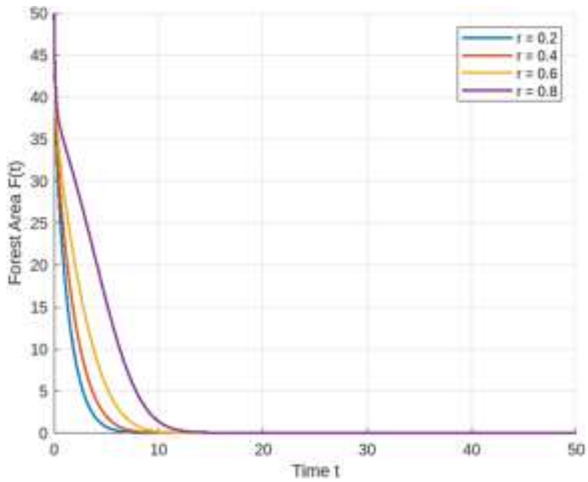


Fig.1: Variation of F(t) for different values of **r**

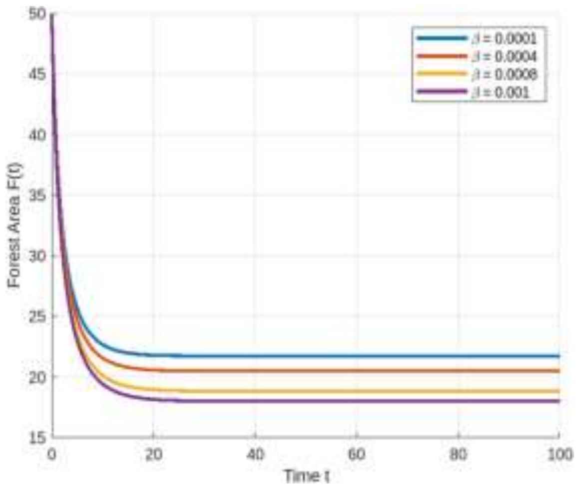
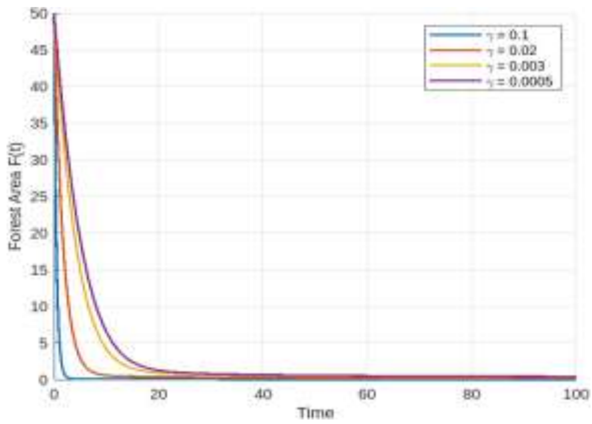


Fig.2: Variation of F(t) for different values of **β**



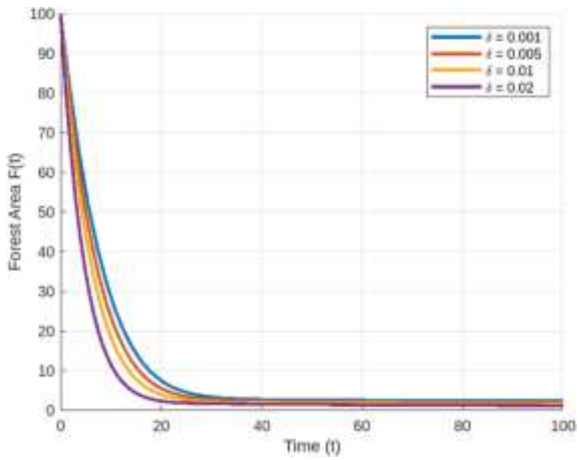


Fig.3: Variation of F(t) for different values of **γ**

Fig.4: Variation of F(t) for different values of **δ**

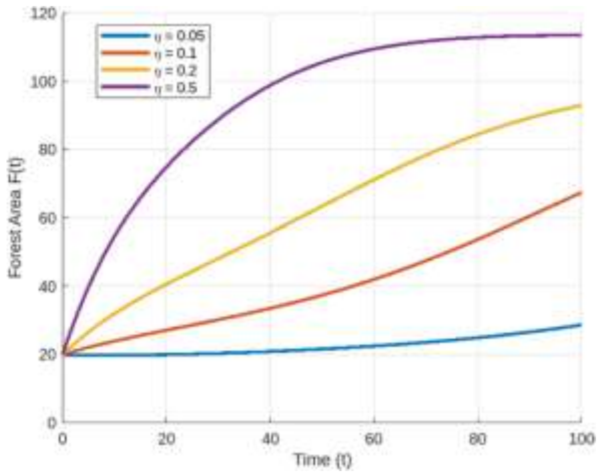


Fig.5: Variation of F(t) for different values of **η**

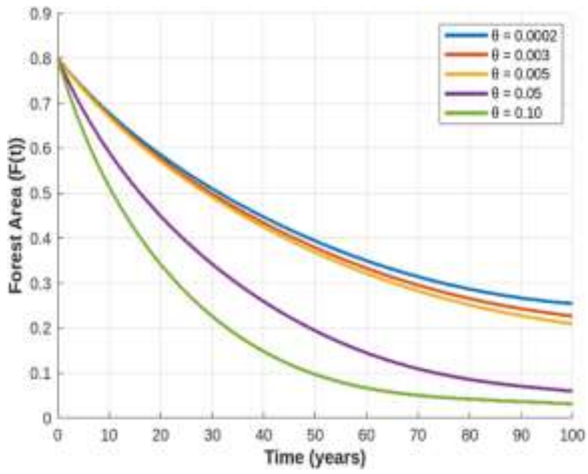


Fig.6: Variation of F(t) for different values of **θ**

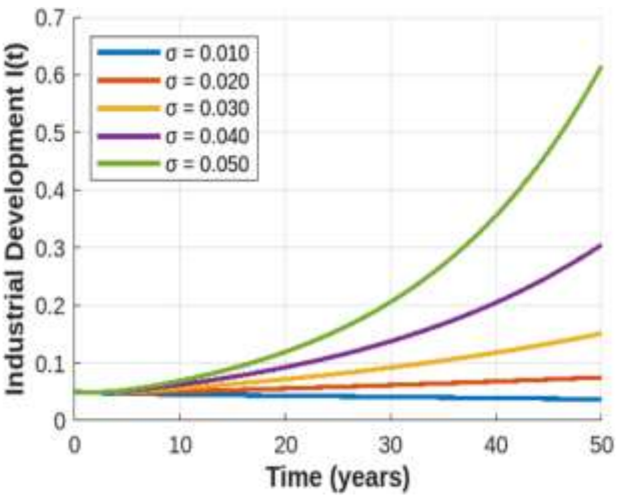


Fig.7: Variation of I(t) for different values of **σ**

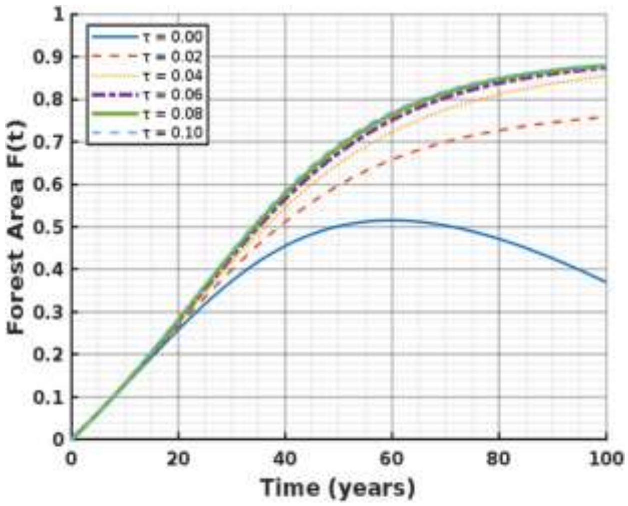


Fig.8: Effect of Industrial Mitigation Rate (**τ**) on Forest Area Dynamics

RESULT AND DISCUSSION

The outcomes of the mathematical model highlight a crucial interaction between different human-induced and natural elements that impact forest dynamics. Numerical simulations conducted with MATLAB's ODE45 solver reveal a rapid decline in forest biomass due to increasing population density, industrial growth, and socio-economic activities. In particular, factors such as the rate of forest degradation caused by industrial activities (γ), the incidence of forest fires (δ), and socio-economic pressures (θ) play a significant role in the depletion path of forest resources. The model indicates that rising values of these parameters result in a steeper decline in forest area over time. On the other hand, reforestation and plantation initiatives, characterized by the parameter η, positively contribute to the restoration of forest biomass. The simulation results demonstrate that an increase in η helps to mitigate forest depletion and has the potential to stabilize forest density, provided there are adequate measures to relieve industrial and socio-economic pressures. Moreover, the interior equilibrium analysis reveals that with a proper mix of conservation measures and the regulation of harmful influences, the system can reach a stable state where forest resources remain at sustainable levels. All associated eigenvalues with this equilibrium are negative, indicating a stable convergence. This suggests that if effective forest conservation actions are taken, the ecosystem can self-regulate and withstand minor disruptions. The interpretation of these results emphasizes the urgent requirement for comprehensive environmental policies. This model offers a solid theoretical basis that reflects the dynamics of forest depletion amid real-world challenges, such as overpopulation and economic advancement. Importantly, reforestation not only has a corrective role but also serves a preventative purpose by mitigating adverse effects from other sectors. Additionally, the model reinforces that uncontrolled industrialization and poor socio-economic regulations expedite the deterioration of forest resources. Notably, population density indirectly contributes to forest degradation by driving both direct consumption and increased industrial and economic activities. The boundedness and positivity of the solutions ensure that the model is relevant and appropriate for long-term forecasts. From a sustainability viewpoint, these findings call for a transition towards equitable development that incorporates afforestation initiatives, manages industrial expansion, and addresses rural communities' reliance on forests. These conclusions resonate with the United Nations' Sustainable Development Goals (especially SDG 13 and SDG 15) and offer practical guidance for policymakers striving to achieve environmental sustainability alongside economic and social progress. Overall, this model acts as a vital analytical resource to inform future conservation efforts and underscores the immediate need for cross-sectoral interventions to avert irreversible ecological harm.

CONCLUSION

This research presents a detailed nonlinear dynamical model that accounts for the various factors contributing to forest depletion as a result of human population growth, industrial progress, reforestation initiatives, forest fires, and socio-economic pressures. By creating a set of six interconnected differential equations, the model effectively illustrates how intricate ecological processes and human activities interplay over time. The analysis of positivity and boundedness affirms the biological relevance of the model, while the stability analysis reveals the circumstances that allow for the sustainable preservation of forest resources. Numerical simulations provide essential insights into how different levels of critical parameters such as industrial impact, socio-economic strain, and reforestation initiatives can markedly affect the long-term stability of forest ecosystems. The findings strongly suggest that without active conservation efforts, forest degradation is unavoidable. Nevertheless, the results also indicate that strategic actions like bolstering reforestation initiatives, controlling industrial growth, reducing forest fires, and alleviating socio-economic pressures can contribute to achieving a stable ecological balance. Furthermore, the stability of the interior equilibrium point shows the system's capacity to sustain forest biomass at viable levels when conditions are favorable. This indicates that a balance between environmental protection and developmental requirements is feasible through informed policymaking and community-led efforts. In summary, the model emphasizes the critical need for a comprehensive approach to forest conservation. It acts as a useful decision-support resource for environmental planners, policymakers, and researchers who seek to formulate long-term strategies that align with sustainable development objectives. By utilizing mathematical modeling to tackle real-world challenges, this study strengthens the significance of interdisciplinary collaboration in combating environmental degradation and highlights pathways to safeguard forest ecosystems for future generations.

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