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Directed Power Graphs Of Non-Coprime Cyclic Group Products

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Abstract
In this paper, we investigate the directed power graph of the direct product of cyclic groups, Z_m and Z_n where m and n are not relatively prime. We conduct a detailed structural analysis of the graphs $G(Z_{p\times p})$, $G(Z_{p\times 2p})$and $G(Z_{p\times p^2})$. Here we are utilizing the algebraic properties of Z_n and Z_m, where m and n are coprime. The study focuses on how the lack of coprimality influences the connectivity and hierarchical structure of these directed power graphs.
Keywords: *Cyclic group, Direct product, Coprime, Degree, Centre of attraction, Petals.*

1 INTRODUCTION

In this paper, we discussed the directed power graph of the direct product of cyclic groups [3], Z_m and Z_n, where *m* and *n* are not relatively prime [4]. Here, we focused on studying the structural properties of certain classes of these digraphs [1] [2] using the properties of Z_m and Z_n.
Let *m* and *n* be two positive integers that are not relatively prime. Consider the direct product $Z_m \times Z_n$ of Z_m and Z_n. Since *m* and *n* are not relatively prime, $Z_m \times Z_n$ is not a cyclic group [3]. Here, we discuss the directed power graph [5] of $Z_m \times Z_n$ denoted by $G(Z_{m\times n})$. Two distinct vertices (x, y) and (u, v) of $G(Z_{m\times n})$ are joined by an arc from (x, y) to (u, v) if and only if (u, v) belongs to the cyclic subgroup generated by (x, y).

For example, Figure 1 shows $G(Z_{2\times 4})$.

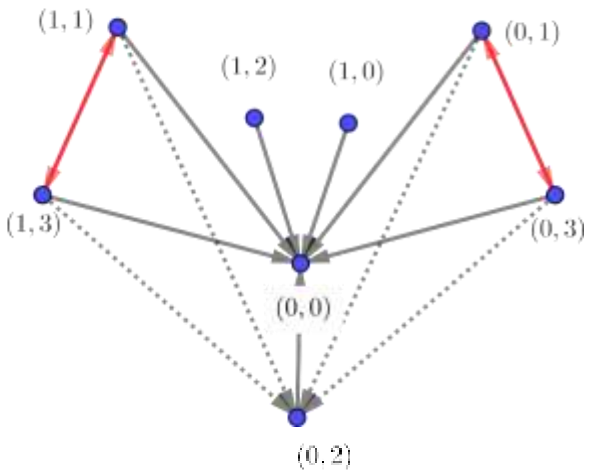

Figure 1: $G(Z_{2\times 4})$

If we consider $G(Z_{m\times n})$ the following are some immediate observations:

- $od((0, 0)) = 0$.
- $id((0, 0)) = mn - 1$
- $od((a, b)) = O((a, b)) - 1$, where $O((a, b))$ is the order [3] of (a, b) in the group $Z_m \times Z_n$.
- If $g.c.d.(x, n) = 1$, then there exist arcs from $(0, x)$ to $(0, y)$, for every $0 \leq y \leq n - 1$.

2 Structural Properties of $G(Z_{p\times p})$

Let *p* be a prime number. Now $G(Z_{p\times p})$ has a particular structure. Let us consider these digraphs as *flowers* with *petals*. A *petal* represents a spanning subdigraph of a collection of vertices in $G(Z_{p\times p})$. $G(Z_{p\times p})$ contains $p + 1$ *petals* with the centre as the vertex $(0, 0)$. Each of these $p + 1$ *petals* contains $p - 1$ vertices other than $(0, 0)$ and they are adjacent to each other or they are reachable from one another.

Let us denote these *petals* by $P, P_0, P_1, P_2, \cdots, P_{p-1}$. Here, P is the *petal* with vertices $(0, 1), (0, 2), \cdots, (0, p-1)$. Now P_i, $i = 0, 1, \cdots, p-1$ are *petals* which are the spanning subdigraph of the vertices in the cyclic subgroup generated by $(1, i)$ for $i = 0, 1, \cdots, p-1$ in $G(Z_{p\times p})$. Figure 2 shows $G(Z_{2\times 2})$ and $G(Z_{3\times 3})$ and Figure 3 shows $G(Z_{5\times 5})$.

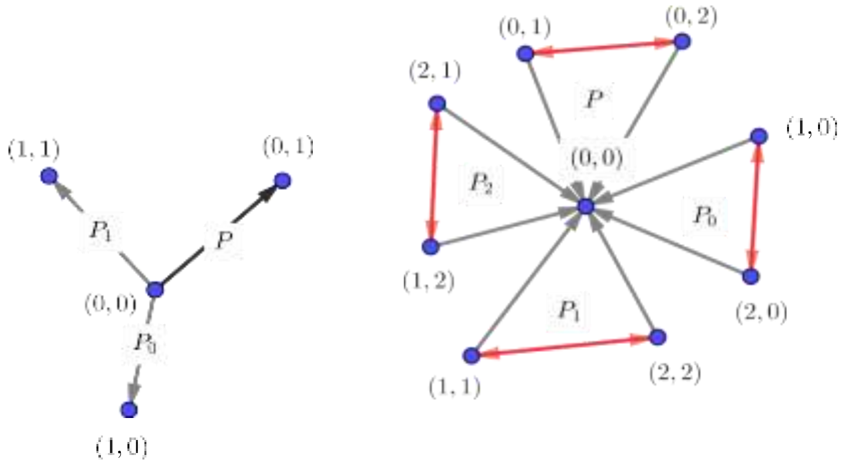

Figure 2: G $(Z_{2\times 2})$ and $G(Z_{3\times 3})$ with *petals*

Theorem 2.1. *Let* (x, y), $0 < x, y < p$ *be a vertex of* $G(Z_{p\times p})$. *Then* $(x, y) \in P_i$, *if and only if* $y \equiv ix(\text{mod}\,p)$, $i = 0, 1, 2, \cdots, p-1$.

Proof. Suppose $(x, y) \in P_i$, then by the definition of P_i, (x, y) and $(1, i)$ are adjacent to each other

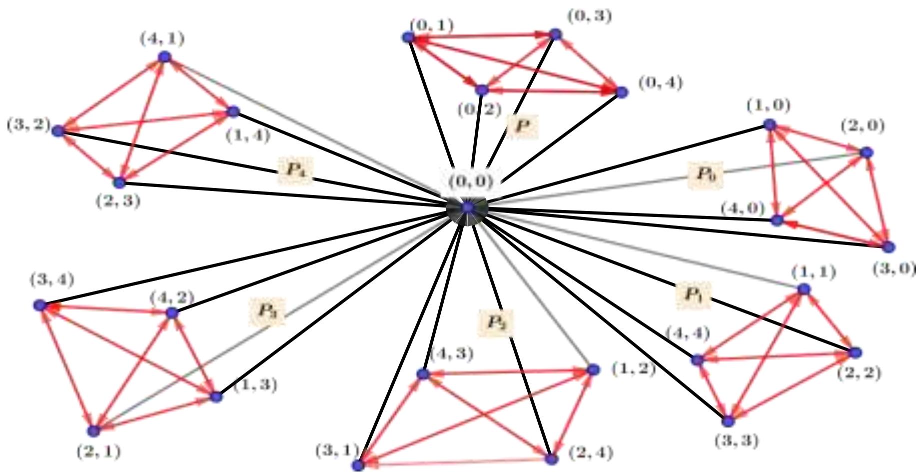

Figure 3: $G(Z_{5\times 5})$ with *petals*

Thus, in $Z_p \times Z_p$, for some positive integer r,

$$\begin{aligned} (x, y) = r(1, i) \quad &\Rightarrow \quad x = r;\ y = ri \text{ in } Z_p \\ &\Rightarrow\Rightarrow \quad y = xi \text{ in } Z_p \\ &\qquad\quad y \equiv xi(\text{mod}\,p). \end{aligned}$$

Conversely, suppose that

$$\begin{aligned} y \equiv xi(\text{mod}\,p) \quad &\Rightarrow \quad y = xi \text{ in } Z_p \\ &\Rightarrow \quad (x, y) = (x, xi) \text{ in } Z_p \times Z_p \\ &\Rightarrow \quad (x, y) = x(1, i) \text{ in } Z_p \times Z_p. \end{aligned}$$

Thus, $(x, y) \in P_i$.

3 Structural Properties of $G(Z_{p\times 2p})$

Let p be an odd prime number. Then $G(Z_{p\times 2p})$ has a particular structure that is different from that of $G(Z_{p\times p})$. In $G(Z_{p\times 2p})$, there exists a vertex $(0, p)$ that has order 2 in the group $Z_p \times Z_{2p}$, which we call the *centre of attraction* of $G(Z_{p\times 2p})$. We denote this vertex by z.

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Theorem 3.1. *Let p be an odd prime. Then the in-degree of the centre of attraction of* $G(Zp\times 2p)$ *is* $p^2 - 1$.

Proof. The vertices in $G(Z_{p\times 2p})$ have an order of either p or $2p$. Consider the vertices $(x, y) \neq (0, 0)$ with $0 \leq x \leq p - 1$, $0 \leq y \leq 2p - 1$ and $y \equiv 1(\text{mod } 2)$ which are different from z. Then the order of these elements in the group $Z_p \times Z_{2p}$ is

$l.c.m\{O(x) \text{ in } Z_p, O(y) \text{ in } Z_{2p}\} = l.c.m\{p, 2p\}$ or $l.c.m\{1, 2p\} = 2p$.

Now, since y is odd, $px \equiv 0(\text{mod } p)$ and $py \equiv p(\text{mod } 2p)$. Therefore, $p(x, y) = (px, py) = (0, p)$ in $Z_p \times Z_{2p}$, and hence these elements, which have order $2p$ in the group $Z_p \times Z_{2p}$ generate $(0, p)$. Thus, there exist arcs from these vertices to $(0, p)$ in $G(Z_{p\times 2p})$. Now there are p choices for x and p choices for y, thus there are p^2 such vertices (x, y). But this includes $z = (0, p)$ also. So the number of vertices G in G whose order $2p$ in the group $Z_p \times Z_{2p}$ is $p^2 - 1$, and these $p^2 - 1$ vertices contribute $p^2 - 1$ to the in-degree of $z = (0, p)$.

Now, let (a, b) be a vertex in $G(Z_{p\times 2p})$ whose order in the group $Z_p \times Z_{2p}$ is p.

Case(i) $a = 0$ and b is even.

Consider the subgroup generated by $(0, b)$. That is,

$<(0, b)>= \{(0, b), (0, 2b), (0, 3b), \cdots, (0, (p-1)b)\}$.

Since b is even, for any positive integer r, rb is not equal to p in Z_{2p}. So an arc does not exist from $(0, b)$ to $(0, p)$.

Case(ii) $a\neq 0$ and b is even.

Consider the subgroup generated by (a, b). That is,

$<(a, b)>= \{(a, b), (2a, 2b), (3a, 3b), \cdots, ((p-1)a, (p-1)b), (pa, pb)\}$

Since b is even, $(pa, pb) = (0, 0)$ in $Z_p \times Z_{2p}$. So the element $(0, p)$ is not in this collection. Hence, there exists no arc from (a, b) to $(0, p)$ in $G(Z_{p\times 2p})$.

Case(iii) $a\neq 0$ and $b = 0$.

Consider the subgroup generated by $(a, 0)$, this subgroup never contains an element of the form $(0, p)$. So there exists no arc from (a, b) to $(0, p)$ in $G(Z_{p\times 2p})$. Therefore, there exists no arc from a vertex whose order in $Z_p \times Z_{2p}$ is p to the vertex $(0, p)$. Hence, the in-degree of *centre of attraction* is $p^2 - 1$.

Note that the spanning subdigraph of the vertices of the form (o, j), for $j = 0, 1, 2, \cdots, 2p - 1$ and $j \neq p$ in $G(Z_{p\times 2p})$ forms a petal which is denoted by P.

Theorem 3.2. *Let p be an odd prime. Then the petals other than P of* $G(Z_{p\times 2p})$ *contain the vertices which are in the cyclic group generated by either the vertex of the form* $(1, i)$ *or the vertex of the form* $(1, p + i)$ *for* $i = 0, 1, \cdots, p - 1$.

Proof. The cyclic subgroup generated by $(1, i)$ contains the elements of the form $k(1, i)$, for $k = 1, 2, \cdots, 2p$.

Case(i) i is odd

Note that $p(1, i) \neq (0, 0)$. Thus,

$<(1, i)>= \{(1, 0), (2, 2i), \cdots, (p-1, (p-i), (0, p), (1, (p+1)i), (2, (p+2)i), \cdots, (0, 0)\}$

having order $2p$ and the spanning subdigraph of $<(1, i)>$ in $G(Z_{p\times 2p})$ forms a *petal*. The generators of this cyclic subgroup are $(1, i), (3, 3i), \cdots, (p-2, (p-2)i), (2, (p+2)i), \cdots, (p-1, (2p-1)i)$ and they are adjacent to each other. The remaining vertices of this *petal* are of order p in the group $Z_p \times Z_{2p}$ and are adjacent to each other by the definition of $G(Z_{p\times 2p})$. So for each odd $i \in \{0, 1, 2, \cdots, p-1\}$, $(1, i)$ generates a *petal*, each of which contains $2p - 1$ vertices.

Case (ii) i is even

Then $<(1, p+i)>= \{k(1, p+i): k = 1, 2, \cdots, 2p\}$ has order $2p$ and these vertices forms a petal. Here $(2, 2i), (4, 4i), \cdots, (p-1, (p-1)i), (1, p+i), (3, p+3i), \cdots$, and $(p-2, p+(p-2)i)$ are the vertices having order $2p$ in $Z_p \times Z_{2p}$ and they are adjacent to each other. The remaining elements of $<(1, p+i)>$ are of order p in $Z_p \times Z_{2p}$ and they are adjacent to each other. So for each even $i \in \{0, 1, 2, \cdots, p-1\}$, $(1, p+i)$ generates a *petal* each of which contains $2p - 1$ vertices.

Now if i is odd, there are $(p-1)/2$ *petals* with $2p - 2$ vertices other than $(0, 0)$ and if i is even, there are $(p+1)/2$ *petals* with $2p - 2$ vertices other than $(0, 0)$. Also, there are $2p - 2$ vertices in the *petal*, P. Thus, the number of vertices in each petal P_i, $i = 0, 1, 2, \cdots, p-1$, and the petal P together with the vertices $(0, 0)$ and $(0, p)$ is

$$\frac{p-1}{2}(2p-2)+\frac{p+1}{1}(2p-2)+(2p-2)+2$$

$$=p^2-2p+1+p^2-1+2p-2+2=2p^2$$

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Since the number of vertices in G ($Z_{p\times 2p}$) is $2p^2$, all the vertices of G ($Z_{p\times 2p}$) are in some unique *petals*.

Note: The *petals* other than P in G ($Z_{p\times 2p}$) contain the vertices which are the elements in the cyclic subgroup generated by $(1, i)$, if i is odd, and by $(1, p + i)$ if i is even and is denoted by P_i, $i = 0, 1, \cdots, p-1$. Thus, we can say that G ($Z_{p\times 2p}$) contains $(p + 1)$ *petals*, and each of these $p + 1$ *petals* contains $2p - 2$ vertices other than $(0, 0)$. Out of these $2p - 2$ vertices of a *petal*, $p - 1$ vertices are of order p and the remaining $p - 1$ vertices are of order $2p$ in the group $Z_p \times Z_{2p}$. The vertices with the same order are adjacent to each other and the vertices with order $2p$ are adjacent to the vertices with order p. Also, there are arcs from the vertices having order $2p$ to the vertex $(0, p)$, *centre of attraction* of G ($Z_{p\times 2p}$).

For example, Figure 4, and Figure 5 show G ($Z_{3\times 6}$), and G ($Z_{5\times 10}$) respectively.

Theorem 3.3. *Let (x, y), $x\neq 0$ be a vertex in G ($Z_{p\times 2p}$). If $(x, y) \in P_i$ then $y \equiv xi (\text{mod} p)$.*
Proof. **Case (i)**: i is odd.
P_i contains the vertices that are in the cyclic subgroup generated by $(1, i)$.
So if (x, y) is in P_i, then for some positive integer $r \neq p$, in $Z_p \times Z_{2p}$,

$$\begin{aligned}(x, y) = r(1, i) &\Rightarrow x = r \text{ in } \mathbb{Z}_p \text{ and } y = ri \text{ in } \mathbb{Z}_{2p}\\ &\Rightarrow x \equiv r(\text{mod}p) \text{ and } y \equiv ri(\text{mod}2p)\\ &\Rightarrow x - r = kp \text{ and } y - ri = k_1 2p, \text{ for some } k, k_1\\ &\Rightarrow y - (x - kp)i = k_1 2p\\ &\Rightarrow y - xi = (2k_1 - ki)p\\ &\Rightarrow y \equiv xi(\text{mod}p).\end{aligned}$$

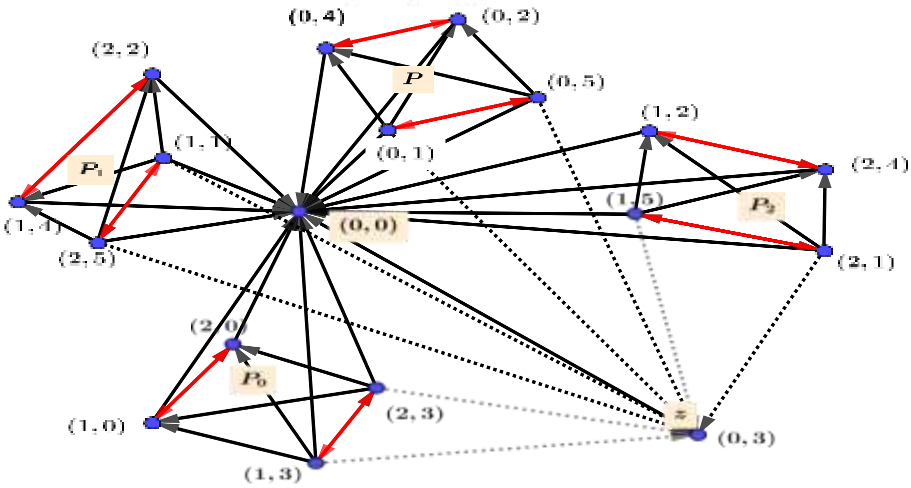

Figure 4: G ($Z_{3\times 6}$)

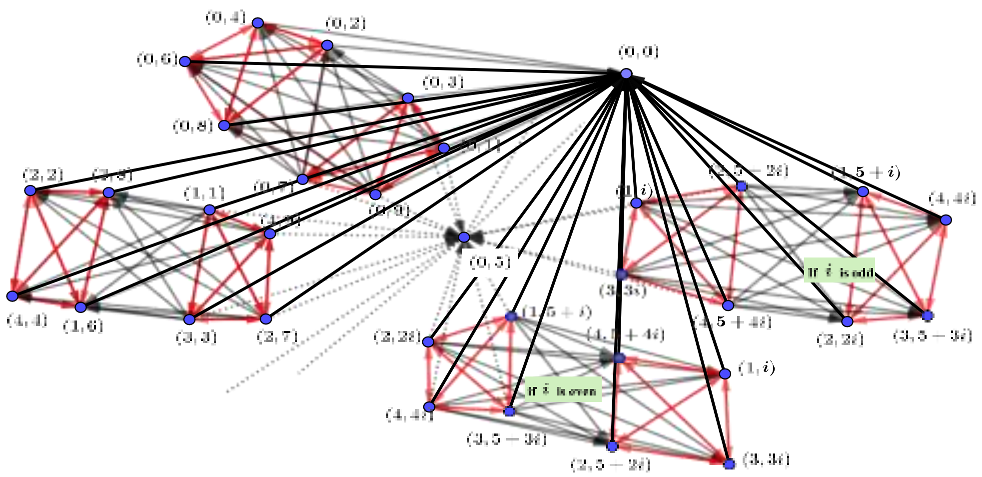

Figure 5: G ($Z_{5\times 10}$)

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Case (ii): i is even.
If (x, y) is in P_i, then for some $r \neq p$ in $Z_p \times Z_{2p}$
$(x, y) = r(1, p + i) \quad \Rightarrow \quad x = r$ in Z_p and $y = r(p + i)$ in Z_{2p}
$\Rightarrow r - x = kp$ and $y - rp - ri = 2k_1p,\ k, k_1 \in Z$
$\Rightarrow r = kp + x$ and $y - r(p + i) = 2k_1p$
$\Rightarrow y - (kp + x)(p + i) = 2k_1p$
$\Rightarrow y - kp^2 - kpi - px - xi - 2k_1p = 0$
$\Rightarrow y - xi \equiv 0(\text{mod} p)$
$\Rightarrow y \equiv xi(\text{mod } p)$.

Thus in both cases $y \equiv xi(\text{mod } p)$.

Theorem 3.4. *Let* $(x, y), x \neq 0$ *be a vertex in* $G(Z_{p\times 2p})$. *If* $y \equiv xi(\text{mod } 2p)$ *or* $y \equiv p + xi(\text{mod } 2p)$, *then* $(x, y) \in P_i$.
Proof. Let us consider two cases.
Case(a): *i* is even
Claim: $(x, y) = r(1, p + i)$, for some integer r.
(i) *x is even.*
Since x is even,
$x(p + i) = xp + xi \equiv xi \equiv y \ (\text{mod } 2p)$.
Also,

$(p + x)(p + i) = (p + x)p + (p + x)i$
$\equiv p + pi + xi \ (\text{mod } 2p)$
$\equiv p + xi \ (\text{mod } 2p)$
$\equiv y \ (\text{mod } 2p)$.

(ii) *x is odd.*
Since x is odd,
$x(p + i) = xp + xi \equiv p + xi \equiv y \ (\text{mod } 2p)$
Also,

$(p + x)(p + i) = (p + x)p + (p + x)i$
$\equiv 0 + pi + xi \ (\text{mod } 2p)$
$\equiv xi \ (\text{mod } 2p)$
$\equiv y \ (\text{mod } 2p)$.
Since $p + x \equiv x(\text{mod } p)$, $(x, y) = r(1, p + i)$ for either $r = x$ or $r = p + x$.
Case(b): *i* is odd.
Claim: $(x, y) = r(1, i)$, for some integer r.
$x(1, i) = (x, xi) \equiv (x, y)$ in $Z_p \times Z_{2p}$.

Also, since i is odd,
$(p + x)(1, i) = (p + x, (p + x)i)$
$= (p + x, pi + xi)$
$= (x, p + xi)$
$= (x, y)$ in $Z_p \times Z_{2p}$.
Thus, $(x, y) = r(1, i)$ for either $r = x$ or $r = p + x$.

4 Structural Properties of $G(Z_{p\times p^2})$

Consider $G(Z_{p\times p^2})$, where p is a prime number. These digraphs contain $p - 1$ vertices of the form $(0, ip)$, where $i = 1, 2, \cdots, p - 1$. We call these vertices *center of attractions* of the digraph and are denoted by z_i, $i = 1, 2, \cdots, p - 1$, whose order in the group $Z_p \times Z_{p^2}$ is p. There exist arcs from all the vertices of $G(Z_{p\times p^2})$ except $(0, 0)$ and the vertices whose order in the group $Z_p \times Z_{p^2}$ is p to these z_i. Now the vertices of the form $i(1,$

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0), $i(1, p)$, $i(1, 2p)$, $\cdots$, $i(1, (p-1)p)$, where $i = 1, 2, \cdots, p-1$ whose order in the group $Z_p \times Z_p{}^2$ is p form $p-1$ *small petals* with $p-1$ vertices. For $j = 1, 2, \cdots, p-1$, $i(1, jp)$ form *small petals*.

Now there exists a *petal* P_0 containing all the vertices of the form $(0, i)$, where $i = 1, 2, \cdots, p^2-1$ except $i = p, 2p, \cdots, (p-1)p$. Also, for $i = 1, 2, \cdots, p-1$, there exists *petals* P_i containing the vertex $(1, i)$. This P_i contains $p(p-1)$ vertices that are adjacent to each other. For a fixed i, vertices in P_i are $(1, i)$, $(1, (1+p)i)$, $(1, (1+2p)i)$, $\cdots$, $(1, (1+(p-1)p)i)$, $(2, 2i)$, $(2, (2+p)i)$, $(2, (2+2p)i)$, $\cdots$., $(p-1, (p-1)i)$, $(p-1, (2p-1)i)$, $(p-1, (3p-1)i)$, $\cdots$, $(p-1, (p-1+(p-1)p)i)$ as elements in $Z_p \times Z_p{}^2$. That is, the vertices are of the form $(1, (1+jp)i)$, $(2, (2+jp)i)$, $\cdots$, $(p-1, ((p-1)+jp)i)$, where, $j = 1, 2, \cdots, p-1$ in $Z_p \times Z_p{}^2$.

That is, for a fixed $k = 1, 2, \cdots, p-1$, the vertices in P_i are of the form $(k, (k+jp)i)$, $j = 1, 2, \cdots, p-1$ in $Z_p \times Z_p{}^2$. For example, Figure 6 shows G $(Z_{3\times 9})$.

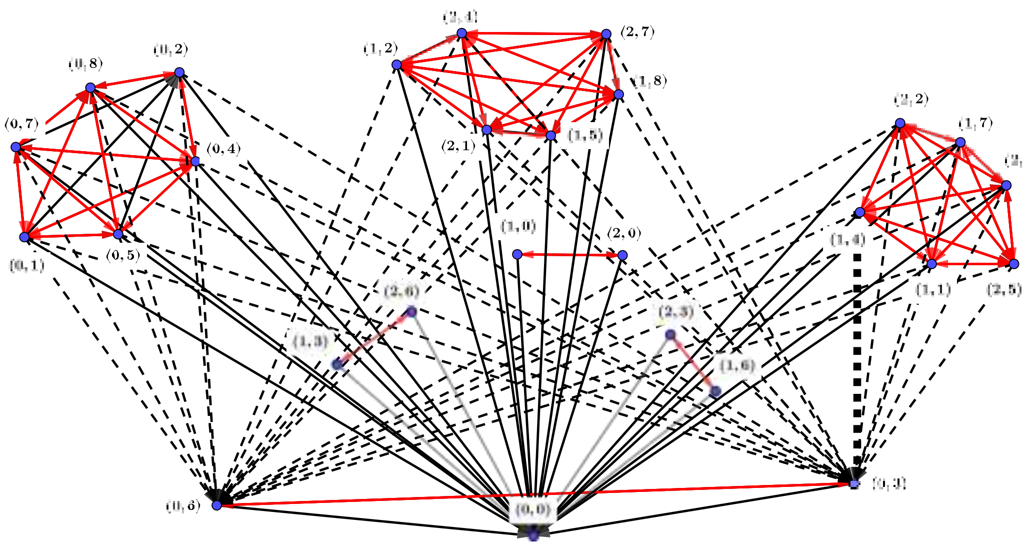

Figure 6 : G $(Z_{3\times 9})$

Theorem 4.1. *Two vertices (a, b) and (a, d) in $G(Z_{p\times p^2})$ are in the same petal P_i if and only if $b-d \equiv 0 (\mathrm{mod}\, p)$.*

Proof. Suppose $(a, b), (a, d) \in P_i$, then $(a, b) = (k, (k + j_1 p)i) \in Z_p \times Z_p{}^2$

$\Rightarrow a = k$ and $b = (a + j_1 p)i \in Z_p{}^2$.

Similarly, $(a, d) \in P_i$. Then, $d = (a + j_2 p)i \in Z_p{}^2$.

So, $b - d = (j_1 - j_2)pi \equiv 0(\mathrm{mod}\, p)$.

Conversely, suppose that (a, b) and (a, d) are in different petals, say, P_s and P_t, respectively.

Then, $b = (a + j_1 p)s$ and $d = (a + j_2 p)t$.

So, $b - d = (as - at) + (j_1 s - j_2 t)p$

$= (s - t)a + (j_1 s - j_2 t)p$

$\equiv (s - t)a \ (\mathrm{mod}\, p)$.

Since $1 \leq s \neq t \leq p - 1$, $(s - t)a \not\equiv 0(\mathrm{mod}\, p)$.

Thus, $b - d \not\equiv 0(\mathrm{mod}\, p)$.

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