International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 2s, 2025
https://www.theaspd.com/ijes.php

Fractional Order Deformable Statistical Measures Skewness And Kurtosis

Hari Pratap[1], Sunit Kumar[2*], Ved Pal Singh[3], Subodh Kumar[4]

[1]Department of Mathematics, P.G.D.A.V. College Evening, University of Delhi-110065,
Email: haripratap@pgdave.du.ac.in
[2*]Department of Mathematics, Motilal Nehru College, University of Delhi-11002, Email: sunit@mln.du.ac.in
[3]Department of Mathematics, Maharaja Agrasen College, University of Delhi-110096, Email: vedpalsingh@mac.du.ac.in
[4]Department of Mathematics, Shyam Lal College, University of Delhi-110032, Email: skumarmath@shyamlal.du.ac.in
*Corresponding author: Sunit Kumar
* Email: sunit@mln.du.ac.in

Abstract:
Memory effects can be easily understood with the help of fractional derivatives [1]. In this paper with the help of Deformable fractional derivative we try to understand the role of deformability on skewness and kurtosis using moment generating function [1]. First, we develop a formula for deformable skewness and kurtosis with the help of deformable fractional derivative using moment generating function. Secondly, we establish a relationship among ordinary skewness, kurtosis to deformable skewness and kurtosis. Lastly, with help of example and a table can see the variation in the statistical measure and draw the conclusions.

Keywords: *Moment generating function, Deformable derivative, Deformable skewness and kurtosis.*

1. INTRODUCTION

Measure of skewness tell us about the lack of symmetry in the distribution, while measure of kurtosis gives information about the peaked-ness or flatness of a distribution. In this paper we show that ordinary skewness and kurtosis is a particular case of deformable skewness and deformable kurtosis. Generalization of the idea of skewness and kurtosis give wide area of insight information about a distribution.

Deformable Fractional Derivative: Let $f(x)$ be a real valued function defined on interval (a, b) for a given number $\alpha,\ 0 \le \alpha \le 1$

$$\lim_{\epsilon \to 0} \frac{(1+\epsilon\alpha^*)\ f(x+\epsilon\alpha) - f(x)}{\epsilon} \tag{1}$$

Where

$$\alpha + \alpha^* = 1$$

If this limit exists, we denote it by $D^\alpha[f(x)]$ [1]-[2].

$$D^\alpha[f(x)] = \alpha\, Df(x) + \alpha^* f(x) \tag{2}$$

Moment Generating Function: Let X be random variable such that for some $h > 0$, the expected value of e^{tx} exists for $-h < t < h$. Then moment generating function of X is defined to be the function $M_X(t) = E[e^{tx}]$, for $-h < t < h$ [1]-[6].

$$M_X(t) = E[e^{tx}] = \sum e^{tx}\ p_X(x) \tag{3}$$

If above series converge then we can say that moment generating function exist.
Deformable α-differentiable of moment generating function is defined as

$$M_X{}^{\alpha}(t) = \sum D^\alpha[e^{tx}]\ p_X(x)$$

$$M_X{}^{\alpha}(t) = \sum\{\alpha\, x\, e^{tx} + (1-\alpha)\, e^{tx}\}\ p_X(x) \tag{4}$$

With the help of equation (4) we can define deformable mean and deformable standard deviation [1].
Deformable Mean

$$\mu^{\alpha} = M_X{}^{\alpha}(0) = \alpha\, E(X) + (1-\alpha) \quad (5)$$

Deformable standard deviation

$$\sigma_X{}^{\alpha} = \alpha\, \sigma_X \quad (6)$$

2. DEFORMABLE COEFFICIENT OF SKEWNESS

Coefficient of skewness tell us about the direction of the variation or the departure from symmetry and is denoted by β_1 and defined as [6]. Whereas, Deformable coefficient of skewness defines using deformable derivative is varying over fractional orders.

As we know that coefficient of skewness

$$\beta_1 = \frac{\mu^2{}_3}{\mu^3{}_2} \quad (7)$$

As we know that

$$\mu_2 = M''_X(0) - (M'_X(0))^2 \quad (8)$$

$$\mu_2^{\alpha} = M^{\alpha.\alpha}{}_X(0) - (M^{\alpha}{}_X(0))^2 \quad (9)$$

$$\mu_3 = M'''_X(0) - 3\, M'_X(0)\, M''_X(0) + 2\,(M'_X(0))^3 \quad (10)$$

$$\mu_3^{\alpha} = M^{\alpha.\alpha.\alpha}{}_X(0) - 3\, M^{\alpha}{}_X(0)\, M^{\alpha.\alpha}{}_X(0) + 2\,(M^{\alpha}{}_X(0))^3 \quad (11)$$

$$\mu_4 = M''''_X(0) - 4\, M'_X(0)\, M'''_X(0) + 6\left(M'_X(0)\right)^2 M''_X(0) - 3(M'_X(0))^4 \quad (12)$$

Here α- deformable mean of fourth order is

$$\mu_4^{\alpha} = M^{\alpha.\alpha.\alpha.\alpha}{}_X(0) - 4\, M^{\alpha}{}_X(0)\, M^{\alpha.\alpha.\alpha}{}_X(0) + 6\left(M^{\alpha}{}_X(0)\right)^2 M^{\alpha.\alpha}{}_X(0) - 3(M^{\alpha}{}_X(0))^4 \quad (13)$$

As we know that using equation (4)

$$M_X{}^{\alpha.\alpha}(t) = \alpha^2 \sum x^2\, e^{tx}\, p_X(x) + 2\,\alpha\,(1-\alpha) \sum x\, e^{tx}\, p_X(x) + (1-\alpha)^2 \sum e^{tx}\, p_X(x) \quad (14)$$

Term by term α- differentiation of eqn. (14) w.r.to t then

$$M_X{}^{\alpha.\alpha.\alpha}(t) = \alpha^2 \sum x^2\, D^{\alpha}[e^{tx}]\, p_X(x) + 2\,\alpha\,(1-\alpha) \sum x\, D^{\alpha}[e^{tx}]\, p_X(x) + (1-\alpha)^2 \sum D^{\alpha}[e^{tx}]\, p_X(x)$$

$$M_X{}^{\alpha.\alpha.\alpha}(t) = \alpha^2 \sum x^2 \{\alpha\, x\, e^{tx} + (1-\alpha)\, e^{tx}\}\, p_X(x) + 2\,\alpha\,(1-\alpha) \sum x \{\alpha\, x\, e^{tx} + (1-\alpha)\, e^{tx}\}\, p_X(x) + (1-\alpha)^2 \sum \{\alpha\, x\, e^{tx} + (1-\alpha)\, e^{tx}\}\, p_X(x)$$

$$M_X{}^{\alpha.\alpha.\alpha}(t) = \alpha^3 \sum x^3\, e^{tx}\, p_X(x) + 2\,\alpha^2\,(1-\alpha) \sum x^2\, e^{tx}\, p_X(x) + \alpha^2\,(1-\alpha) \sum x^2\, e^{tx} p_X(x) + 2\alpha(1-\alpha)^2 \sum x\, e^{tx}\, p_X(x) + (1-\alpha)^3 \sum e^{tx}\, p_X(x) + \alpha(1-\alpha)^2 \sum x e^{tx}\, p_X(x) \quad (15)$$

$$M_X{}^{\alpha.\alpha.\alpha}(0) = \alpha^3 \sum x^3\, e^0\, p_X(x) + 3\,\alpha^2\,(1-\alpha) \sum x^2\, e^0\, p_X(x) + 3\alpha(1-\alpha)^2 \sum x\, e^0\, p_X(x) + (1-\alpha)^3 \sum e^0\, p_X(x)$$

$$M_X{}^{\alpha.\alpha.\alpha}(0) = \alpha^3\, E(X^3) + 3\alpha^2\,(1-\alpha)\, E(X^2) + 3\alpha(1-\alpha)^2\, E(X) + (1-\alpha)^3 \quad (16)$$

As we know that α-deformable moment of third order is

$$\mu_3^{\alpha} = M^{\alpha.\alpha.\alpha}{}_X(0) - 3\, M^{\alpha}{}_X(0)\, M^{\alpha.\alpha}{}_X(0) + 2\,(M^{\alpha}{}_X(0))^3 \quad (17)$$

After solving we get

$$\mu_3^\alpha = \alpha^3 \left[E(X^3) + 2\left(E(X)\right)^3 - 3\,E(X)E(X^2)\right] + 3\,\alpha^2(1-\alpha)\,E(X) \quad (18)$$

Deformable coefficient of skewness is defined as

$$\beta_1{}^\alpha = \frac{(\mu_3^\alpha)^2}{(\mu_2^\alpha)^3} \quad (19)$$

$$\beta_1{}^\alpha = \frac{[\alpha^3\left[E(X^3)+2\left(E(X)\right)^3-3\,E(X)E(X^2)\right]+3\,\alpha^2(1-\alpha)\,E(X)\,]^2}{[\alpha^2\,E(X^2)+2\,\alpha\,(1-\alpha)E(X)+(1-\alpha)^2]^3} \quad (20)$$

It shows the fractional order $(\alpha = 1)$, dependency of coefficient of skewness on deformable coefficient of skewness.

3. DEFORMABLE COEFFICIENT OF KURTOSIS

Whereas, Deformable coefficient of kurtosis defines using deformable derivative and varying over fractional orders.

As we know that the coefficient of kurtosis is

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \quad (21)$$

From equation (15)

$$M_X{}^{\alpha.\alpha.\alpha}(t) = \alpha^3 \sum x^3\,e^{tx}\,p_X(x) + 2\,\alpha^2\,(1-\alpha)\sum x^2\,e^{tx}\,p_X(x) + \alpha^2\,(1-\alpha)\sum x^2\,e^{tx}p_X(x) + 2\alpha(1-\alpha)^2\sum x\,e^{tx}\,p_X(x) + (1-\alpha)^3\sum e^{tx}\,p_X(x) + \alpha(1-\alpha)^2\sum xe^{tx}\,p_X(x)$$

Term by term α- differentiation of above equation w.r.to t then

$$M_X{}^{\alpha.\alpha.\alpha.\alpha}(t) = \alpha^3\sum x^3\,D^\alpha[e^{tx}]p_X(x) + 3\,\alpha^2\,(1-\alpha)\sum x^2\,D^\alpha[e^{tx}]p_X(x) + 3\alpha(1-\alpha)^2\sum x\,D^\alpha[e^{tx}]p_X(x) + (1-\alpha)^3\sum D^\alpha[e^{tx}]p_X(x)$$

After solving the above equation at $t = 0$ we get

$$M_X{}^{\alpha.\alpha.\alpha.\alpha}(0) = \alpha^4\,E(X^4) + 4\,\alpha^3\,(1-\alpha)E\,(X^3) + 6\,\alpha^2\,(1-\alpha)^2\,E(X^2) + 3\alpha\,(1-\alpha)^3E(X) + (1-\alpha)^4$$

$$\mu_4^\alpha = M_X{}^{\alpha.\alpha.\alpha.\alpha}(0) - 4\,M_X{}^\alpha\,(0)M_X{}^{\alpha.\alpha.\alpha}(0) + 6\left(M_X{}^\alpha(0)\right)^2 M_X{}^{\alpha.\alpha}(0) - 3\,(M_X{}^\alpha(0))^4$$

On solving above equation, we get

$$\mu_4^\alpha = \alpha^4\,[E(X^4) - 3\{E(X)\}^4 + 6\,\{E(X)\}^2E(X^2) - 4\,E(X)E(X^3)] - \alpha\,(1-\alpha)^3\,E(X) - 9\,(1-\alpha)^4$$

Deformable coefficient of kurtosis is defined as

$$\beta_2^\alpha = \frac{(\mu_4^\alpha)}{(\mu_2^\alpha)^2} \quad (22)$$

$$\beta_2^\alpha = \frac{\alpha^4\,[E(X^4) - 3\{E(X)\}^4 + 6\,\{E(X)\}^2E(X^2) - 4\,E(X)E(X^3)] - \alpha\,(1-\alpha)^3\,E(X) - 9\,(1-\alpha)^4}{[\alpha^2\,E(X^2) + 2\,\alpha\,(1-\alpha)E(X) + (1-\alpha)^2]^2}$$

(23)

4. EXAMPLE

For the following data compute the coefficient of skewness and kurtosis [7].

X:	4.5	14.5	24.5	34.5	44.5	54.5	64.5	74.5	84.5	94.5
f:	1	5	12	22	17	9	4	3	1	1

The following are the calculated values on the basis of given data.
$E(X) = -0.4,$ $E(X^2) = 2.99,$ $E(X^3) = -0.08,$ $E(X^4) = 27.63$
$\beta_1 = 0.504,$ $\beta_2 = 3.782$

Table 4.1 for deformable skewness and kurtosis

Fractional order (α)	μ_2^α	μ_3^α	μ_4^α	Deformable skewness($\beta_1{}^\alpha$)	Deformable kurtosis (β_2^α)
0.1	0.76790	-0.00742	-5.87271	0.00012	-9.95931
0.2	0.63160	-0.01136	-3.59697	0.00051	-9.01680
0.3	0.59110	0.01566	-1.87435	0.00119	-5.36448
0.4	0.64640	0.10112	-0.35627	0.03786	-0.85267
0.5	0.79750	0.27250	1.35597	0.14640	2.13202
0.6	1.04440	0.55728	3.71127	0.27261	3.40243
0.7	1.38710	0.98294	7.20863	0.36202	3.74660
0.8	1.87560	1.57696	12.39724	0.40872	3.71975
0.9	2.35990	2.36682	19.87640	0.42624	3.56903
1	2.9	3.38000	30.29	0.42739	3.38873

5. CONCLUSION

Skewness and kurtosis are a particular case of deformable skewness and kurtosis for $\alpha = 1$.To verify this result we take an example and calculate value of $\mu_2^\alpha, \mu_3^\alpha, \mu_4^\alpha$ for varying fractional orders for $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$ and also calculate the varying values of deformable skewness and kurtosis. On the basis of table 4.1 we can see that deformable skewness and kurtosis are approximately very close at $\alpha = 1$. Therefore, we can conclude that skewness and kurtosis are particular case of deformable skewness and kurtosis.

REFERENCES:

[1] Hari Pratap, Sunit Kumar, Subodh Kumar, Gajraj Singh, Ved Pal Singh, Memory Effect of Deformable Mean and Deformable Standard Deviation, Communications on Applied Nonlinear Analysis, ISSN:1074-133X, Vol 32 No. 8s (2025).
[2] Fahed Zulfeqarr, Amit Ujlayan, Priyanka Ahuja, A New Fractional Derivative and Its Fractional Integral with Some Applications, arXiv:1705.00962v1 [math.CA] 26 April 2017.
[3] R. Khalil, M. AL Horan, A. Yousef, & M. Sababheh, A New Definitions of Fractional Derivative. Elsevier, Journal of Computational and Applied Mathematics, 2014, 264, 65-70.
[4] Miller, Irvin & Miller, Marylee's (2006). John E. Freund's, Mathematical Statistics with Applications (7th ed.). Pearson Education Asia.
[5] Jay L. Devore, & Kenneth N. Berk (2007), Mathematical Statistics with Applications, Thomson Brooks/ Cole.
[6] Robert V. Hogg, Alen Craig, Joseph W. McKean (2011), Introduction to Mathematical Statistics, Sixth Edition, Pearson.
[7] S.P. Gupta, Statistical Method, Sultan Chand & Sons, 2013.
[8] K.B. Oldham, & J. Spanier, The Fractional Calculus, New York, Academic Press, 1974.
[9] Udita N. Katugampola, New Fractional Derivative with Classical Properties, Journal of The American Mathematical Society, vol. 00, No. 0, pp. 000-000, arXiv:1410.6535v2 [math.CA] 8 Nov. 2014.
[10] Udita N. Katugampola, A new Fractional Derivative with Classical Properties, Journal of American Mathematical Society volume, 2014.

[11] Tarasov, Vasily E., No Violation of The Leibniz Rule, No Fractional Derivative, Elsevier, Communication to Nonlinear Science and Numerical Simulation, 18 (2013), pp. 2945-2948.
[12] Tarasov, Vasily E., On Chain Rule of Fractional Derivatives, Elsevier, Communication to Nonlinear Science and Numerical Simulation, 30 (2016), pp. 1-4.
[13] Abdeljawad, T., On Conformable Fractional Calculus, Elsevier, Journal of Computational and Applied Mathematics, 279 (2015), pp. 57-66.