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Edge-Magic Labeling Of Some Graphs

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Abstract. *The Edge Magic constants of the star graph $k_{1,n}$ are discussed in this study. When p is odd, we determine the minimum and maximum edge magic constants for star, bi-star, and cyclic graphs. We also determine the star graph's midpoint based on the minimum and maximum. This paper discusses the labeling method of cyclic graphs using the symmetric group approach.*

Key Words: *Edge Magic Labeling – Star Graph – Bi–Star Graph – Cyclic – Maximum Edge-Magic constant – Minimum Edge-Magic constant*

Mathematical Subject Classification (2010): 05C78

1. INTRODUCTION

Three types of labeling were introduced and characterized by Jogef Sedlacek [18] in 1963. He dubbed them α, β and ρ labeling magic labeling. Alex Rosa [1] first proposed Edge-magic labeling, a graph labeling technique, in 1967. Conway proposed the idea of maximum and minimum magic constants for a number of graphs, and Rosa's work expanded on his original concepts and laid the groundwork for future studies on magic constants.

Sedlacek's theorem also introduced the idea of the magic constant, which has to do with magic labeling. According to the Rigel's conjecture (1963), there is a magic label for every three. Vertex magic labeling was first proposed by Joseph A. Gallain [7] in 1989. The Survery of Gallian (1998) a thorough analysis of magic constants and labeling.

2. BASIC DEFINITIONS

In this section, we recall some graph definitions relevant to this paper.

Definition 2.1 [2] (Labeled Graph) A graph G is labeled if it's p-points are distinguished from one another by names suchs as $v_1, v_2, \dots, v_p$.

Definition 2.2 [10] (Star Graph) A graph G is said to be a star graph.Ifthere exist a fixed vertex v(called the center of the Star graph) such that $E = \{uv/u \in V \text{ and } u \neq v\}$. A star graph is said to be an n-Star graph if the number of vertices of the graph is n.

Definition 2.3 [10] (Bi-star Graph) A Bi-Star graph is a graph that is created by connecting the center vertices of two copies of the star graph $k_{1,n}$ by an edge. It is denoted by $B_{n,n}$.

Definition 2.4 [2] (Cyclic Graph) A Cyclic Graph is a graph that consists of a single cycle. In other words some numbers of vertices connected in a closed chain. The cycle graph with n vertices is denoted by C_n

Definition 2.5 (Symmetric Group) The symmetric group defined over any set is the group whose elements are all the bi-junctions from the set to itself.

Definition 2.6 [11] (Edge-Magic Labeling) An edge-magic labeling f is a bijection from $V(G) \cup E(G)$ to the set of integers $\{1,2,\dots,p+q\}$ such that if xy is an edge of G, then $f(x) + f(y) + f(xy) = \lambda$ for some integer constant λ.

3. MINIMUM EDGE-MAGIC CONSTANT

In this subdivision, we provide examples and theorems of the Minimum Edge-Magic Constant of star graphs.

Theorem 3.1 The minimum Edge-Magic constant of star graph $k_{1,n}$ is $S^*_{min} = 2(n+2)$.
Proof. By the definition of Edge-Magic Labeling of star graph, we get
$nf(v_1) + f(v_2) + f(v_3) + \cdots + f(v_p) + a_{12} + a_{13} + \cdots + a_{1p} = nS^*,$ (3.1)
where v_1 is centre of the star graph. If we label the number 1 with the centre of vertices, we obtain
$n + \{2 + 3 + \cdots + p\} + p + 1 + p + 2 + \cdots + p + q = nS^*$. (3.2)
All the remaining vertices are labeled with numbers $\{2,3,\dots,p\}$ and edges are labeled with the numbers $\{p+1, p+2, p+3, \dots, p+q\}$, we arrive
$nS^*_{min} = n + \frac{p(p+1)}{2} - 1 + pq + \frac{q(q+1)}{2}$. (3.3)
Since the graph is star graph $k_{1,n}$, we have
$q = p - 1$. (3.4)
From (3.3) and (3.4), we see $S^*_{min} = 2(n+2)$.
Example 3.2

1. Consider the star graph $K_{1,2}$

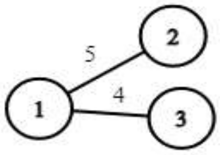

Fig 3.1
the minimum edge-magic constant for above graph Fig. 3.1 is $S^*_{min} = 8$.

2. Consider the star graph $K_{1,5}$

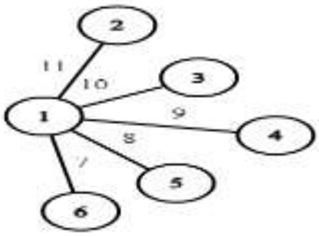

Fig 3.2
the Minimum Edge-magic Constant for above graph Fig. 3.2 is $S^*_{min} = 14$.
Theorem 3.3 The General method to the Edge-Magic Labeling of the star Graph where the magic constant is Minimum.
Proof. Let us consider a Star graph $k_{1,n}$. If v_1 is the centre of the star graph then $v_1 = p - q$ and v_j be any other vertex of the graph then $v_j = p - q - 1 + j$, $j = 2,3,\dots,p$.
Now, the edges are labeled in the following way

$$v_1 + a_{1j} + v_j = (p-q) + (p-q-1+j) + a_{1j}$$
$$\Rightarrow a_{1j} = 2n + 3 - j \ .$$

3. Consider the graph $k_{1,5}$ where $p = 6, q = 5$

Vertex Label - Centre vertex $v_1 = p - q = 1$, Other vertices $v_j = p - q - 1 + j$ where $j = 2,3,\dots,p$, we get $v_2 = 2, v_3 = 3, v_4 = 4, v_5 = 5, v_6 = 6$.
Edge Label - $a_{1j} = 2n + 3 - j$, we secure $a_{12} = 11, a_{13} = 10, a_{14} = 9, a_{15} = 8, a_{16} = 7$.

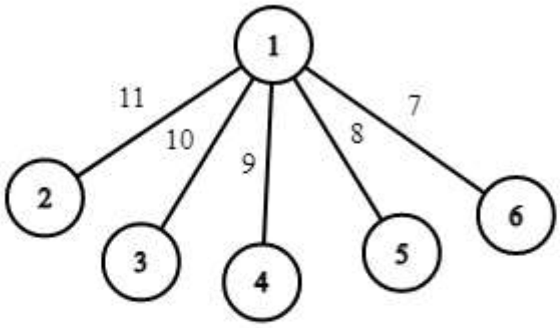

Fig 3.3

Theorem 3.4 If G is a star graph then the maximum edge-magic constant $S^*_{max} = 2(2n+1)$.

Proof. If G is a star graph $k_{1,n}$ by the definition of edge-magic labeling we get the maximum edge-magic constant the vertices of the star graph are labeled with number $\{q+1,\ q+2, \dots, q+p\}$ and edges are labeled with numbers $\{1,2,3,\dots,q\}$ then we get

$nS^*_{max} = nf(v_1) + f(v_2) + \cdots + f(v_p) + a_{12} + a_{13} + \cdots + a_{1p}$. (3.8)

We label the number$(p+q)$ for the centre of the vertex of star graph

$$\begin{aligned} nS^*_{max} &= n(p+q) + \{q+1+q+2+\cdots+q+(p-1)\} + 1+2+\cdots+q \\ &= n(p+q) + (p-1)q + \frac{p(p-1)}{2} + \frac{q(q+1)}{2} \\ &= n(2p-1) + p^2 - 2p + 1\ + p(p-1) \\ &= 2np - n + p^2 - 2p + 1 + p^2 - p \\ &= 2(n+1)^2 + 2n(n+1) - 3(n+1) - n + 1 \\ &= 4n^2 + 2n \end{aligned}$$

$S^*_{max} = 2(2n+1)$.

Examples 3.5

1. Consider the star graph $K_{1,2}$

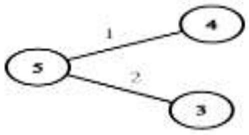

Fig 3.4

the maximum Edge-magic constant for above graph Fig. 3.4 is $S^*_{max} = 10$.

2. Consider the star graph $K_{1,3}$

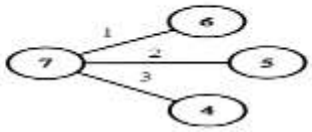

Fig 3.5

the maximum Edge-magic constant for above graph Fig. 3.5 is $S^*_{max} = 14$.

3. Consider the star graph $K_{1,5}$

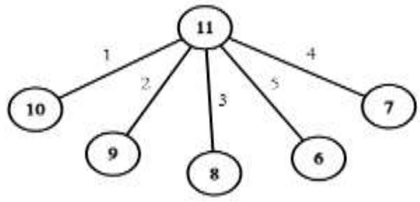

Fig 3.6

the maximum Edge-magic constant for above graph Fig. 3.6 $S^*_{max} = 22$.

Theorem 3.6 If G is a star graph $K_{1,n}$ then there exist a Edge-Magic constant S^*_{mid} between S^*_{min} and S^*_{max} such that $S^*_{mid} = 3(n+1)$

Proof. Given graph G is star graph $K_{1,n}$ the vertices of the graph are labeled with the numbers $\{1,2,3,\dots,p\}$ and edges are labeled with numbers $p+1, p+2, \dots, p+q$, we get

$$nS^*_{mid} = np + \{2+3+4+\cdots+p-1\} + \{p+1+p+2+\cdots+p+q\}$$
$$= np + \left\{\frac{p(p-1)}{2} + pq + \frac{q(q+1)}{2}\right\}$$
$$= np + 2p(p-1)$$
$$= 3n(n+1)$$
$$S^*_{mid} = 3(n+1).$$

Note:- $S^*_{mid} = \frac{S^*_{min}+S^*_{max}}{2} = \frac{2(n+2)+2(2n+1)}{2} = 3(n+1)$.

Examples 3.7

1. Consider the star graph $K_{1,2}$

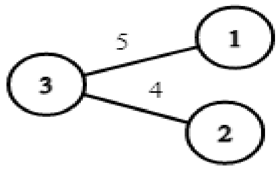

Fig 3.7

there exist a Edge-magic constant above graph Fig. 3.7 is $S^*_{mid} = 9$.

2. Consider the star graph $K_{1,5}$

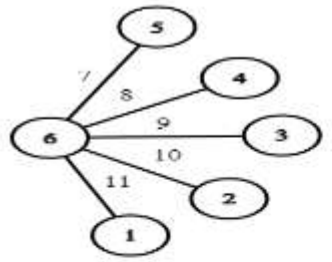

Fig 3.8

there exist a Edge-magic constant above graph Fig. 3.8 is $S^*_{mid} = 18$.

4. THE GENERAL METHOD TO THE EDGE-MAGIC LABELING OF THE STAR GRAPH WHEN THE MAGIC CONSTANT IS MAXIMUM

In this division, we determine the general approach for the general method for Minimum Edge-Magic Constant of star graphs when the magic constant is minimum.

4.1 Method of labeled on $k_{1,n}$

Given graph is a star graph $k_{1,n}$.

If v_i is centre of the star graph then the vertices of the graph are labeled by the following way. $v_i = p+q+1-i$ where v_i is the centre of the graph and $i = 1,2,\dots,p$.

Let v_j be any other vertex of the graph then $v_j = p+q+1-j, j = 2,3,\dots,p$. (4.1)

Now, the edges are labeled by the following way

$v_i + a_{ij} + v_j = 2(p+q+1) - (i+j) + a_{ij}$

$\Rightarrow 2(p+q+1) - (i+j) + a_{ij} = 2(2n+1)$

$\Rightarrow a_{ij} = 2(2n+1) - 2(p+q+1) + (i+j)$. (4.2)

1. Consider the star graph $K_{1,4}, p = 5, q = 4$.

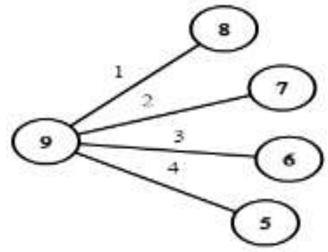

Fig 4.1

Vertex Label - First the centre Vertex $v_i = p + q + 1 - i, v_1 = 9, v_2 = 8$.

Also, similarly we get $v_3 = 7, v_4 = 6, v_5 = 5$

Edge Label - $a_{ij} = 2(2n+1) + i + j - 2(p + q + 1)$

$$= 2(2n+1) + i + j - 2(n + 1 + n + 1)$$
$$= 4n + 2 + i + j - 4n - 4.$$

$a_{ij} = i + j - 2$, $a_{12} = 1$, $a_{13} = 2$, we get the label.

Theorem 4.1 If G is a bi-star graph then the minimum edge-magic constant of the bi-star graph G is $S^{**}_{min} = \frac{deg\, v_1 + p\, deg\, v_2 + 2q(q+1) - 1}{q}$.

Proof. Given graph G is bistar graph, let v_1 and v_2 be the centre of the bi-star graph then by the definition of Edge-Magic Labeling

$$deg\, v_1\, f(v_1) + deg\, v_2\, f(v_2) + f(v_3) + f(v_4) + \cdots + f(v_p) + a_{12} + a_{13} + \cdots + a_{1p} = qS^{**}_{min} \tag{4.4}$$

Now, the centre of the one star is v_1 which is labeled with the number 1 and the centre of the other vertex is v_2 which is labeled with the number p and the remaining vertices are labeled with the numbers $\{2,3,\dots,p-1\}$ and edges of the graph are labeled with the numbers $\{p+1, p+2, \dots, p+q\}$. So, we have

$$deg\, v_1 + p\, deg\, v_2 + \{2 + 3 + \cdots + p - 1\} + \{(p+1) + (p+2) + \cdots + (p+q)\} = qS^{**}_{min}$$

$$deg\, v_1 + p\, deg\, v_2 + \frac{p(p-1)}{2} - 1 + pq + \frac{q(q+1)}{2} = q\, S^{**}_{min}$$

$$q\, S^{**}_{min} = deg\, v_1 + p\, deg\, v_2 + 2q(q+1) - 1$$

$$S^{**}_{min} = \frac{deg\, v_1 + p\, deg\, v_2 + 2q(q+1) - 1}{q}.$$

Note: Suppose the bi-star graph is denoted $B_{n,m}$ then the Edge-magic constant

$$S^{**}_{min} = \frac{(n+1) + (m+1)p + 2q(q+1) - 1}{q}$$

Example 4.2

1. Consider the bi-star graph $B_{2,2}$,

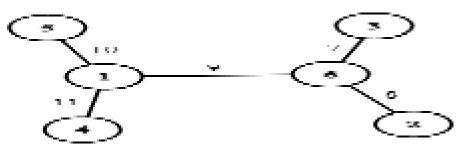

Fig 4.2

$deg\, v_1 = 3, deg\, v_2 = 3$ then above graph Fig. 4.2 is $S^{**}_{min} = 16$.

2. Consider the G is bistar graph $B_{3,3}$

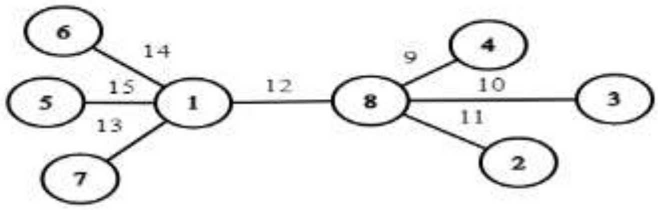

Fig 4.3
then the Edge-magic Constant above graph Fig. 4.3 is $S^{**}_{min} = 21$

Theorem 4.3 If G is bi-star graph then the maximum Edge-magic constant of the graph

$$\frac{(q+1)\,deg\,v_1 + (p+q)\,deg\,v_2}{q} + \frac{pq - 2 + 2q(p-2) + q(q+1)}{2q}. \qquad (4.7)$$

Proof. Given graph G is bi-star graph let v_1, v_2 be the centre of the bi-star graph.The vertices of the bi-star graph are labeled with the number $\{1,2,3,\dots,p\}$ and edges of the star graph are labeled with the numbers $\{p+1, p+2, \dots, p+q\}$. So, we have

$$deg\,v_1\,f(v_1) + deg\,v_2\,f(v_2) + f(v_3) + f(v_4) + \cdots + f(v_p) + a_{12} + a_{13} + a_{14} + \cdots + a_{1\,deg\,v_i} + \cdots + a_{2p} = S^{**}_{max}$$

$$S^{**}_{max} = (q+1)\,deg\,v_1 + (p+q)\,deg\,v_2 + \{q+2+q+3+\cdots+q+(p-1)\} + \{1+2+3+\cdots+q\}$$

$$S^{**}_{max} = \frac{(q+1)\,deg\,v_1 + (p+q)\,deg\,v_2}{q} + \frac{p}{2} - \frac{1}{q} + p - 2 + \frac{q+1}{2}$$

$$= \frac{(q+1)\,deg\,v_1 + (p+q)\,deg\,v_2}{q} + \frac{pq - 2 + 2q(p-2) + q(q+1)}{2q}.$$

Theorem 4.4 If G is a cyclic graph with odd number of vertices then the Maximum Edge-Magic constant $E_{max} = \frac{7p+3}{2}$.

Proof. By the definition of Edge-Magic Labeling $f(u) + f(v) + f(uv) = k$, where $f(u), f(v)$ are the labels of the vertices u, v in orderly and $f(uv)$ be the label of Edge uv of the cyclic graph. Now,

$$f(v_1) + a_{12} + f(v_2) = E_{max}$$
$$f(v_2) + a_{23} + f(v_3) = E_{max}$$
$$\vdots$$
$$f(v_p) + a_{p1} + f(v_1) = E_{max}$$

Add this we get, $2[f(v_1) + f(v_2) + \cdots + f(v_p)] + [a_{12} + a_{23} + a_{34} + \cdots + a_{p1}]$.(4.8)

The vertices are labeled with numbers $\{p+1, p+2, \dots, p+q\}$ and edges are labeled with the number $\{1,2,\dots,p\}$. Then we get,

$qE_{max} = 2[\,p+1+p+2+\cdots+p+q] + [1+2+3+\cdots+p]$

$qE_{max} = 2pq + q(q+1) + \frac{p(p+1)}{2}$,

Since G is cyclic graph, we have $p = q$ and from above we arrive

$$E_{max} = \frac{7q+3}{2}.$$

Example 4.5

1. If G is cyclic graph with 5-points

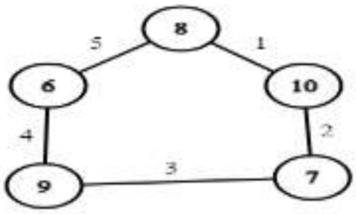

Fig 4.4

then the maximum Edge-magic constant is $E_{max} = 19$

2. If G is cyclic graph with 7-points

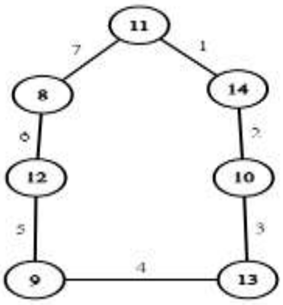

Fig 4.5

then the maximum Edge-magic constant is $E_{max} = 26$

5. THE GENERAL METHOD FOR EDGE-MAGIC LABELING OF CYCLIC GRAPH

In this segment, the general approach for determining Minimum Edge-Magic Constant of cyclic graphs is provide.

5.1 Method for Edge-Magic Labeling on $p = q$

The graph G is cyclic $(p = q)$ also p is odd. The vertex of the cyclic graph are labeled with the numbers $\{1,2,3, \dots p\}$ and edge are labeled with the numbers $\{p + 1, p + 2, \dots, p + q\}$ as the follows,

First the Edges are labeled by the following way

$$a_{12} = 1, a_{23} = 2, a_{34} = 3, \dots, a_{p1} = p.$$

The vertices of the cyclic graph are labeled with numbers as the following way

$$\begin{pmatrix} p+1 & p+2 & \dots & \frac{2p+q+1}{2} & \dots & p+q \\ \frac{2p+q+1}{2} & \frac{2p+q+1}{2}+1 & \dots & p+q & \dots & \frac{2p+q+1}{2}-1 \end{pmatrix}$$

i.e use the symmetric group method.

The numbers $\{p + 1, p + 2, \dots, p + q\}$ are arranged with first row of the symmetric group and we start $\frac{2p+q+1}{2}$ in the first term of the second row continue this way the term become $p + q$, we stop next start with $p + 1$

Example 5.1 Suppose the cyclic graph with 5-vertices

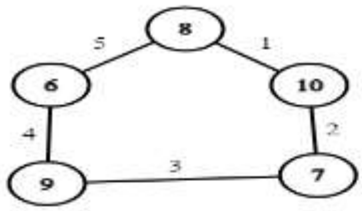

Fig 5.1

having $E_{max} = 19$ the Edge-magic label of the graph by the following way. The edge label.

$$a_{12} = 1, a_{23} = 2, a_{34} = 3, a_{45} = 4, a_{51} = 5$$

And vertex label

$$\begin{pmatrix} 6 & 7 & 8 & 9 & 10 \\ 8 & 9 & 10 & 6 & 7 \end{pmatrix}$$

The maximum sum of 18. Which later to the edge a_{12}

Theorem 5.2 If G is a cyclic graph with p-point and p is odd. Then the minimum Edge-magic constant $E_{min} = \frac{5p+3}{2}$

Proof. By the definition of Edge-magic constant. we get the minimum value when the edges are labeled with the numbers $\{1,2,\dots,q\}$ and vertices are labeled with the numbers $\{q+1,q+2,\dots,q+p\}$

$$qE_{min} = 2[1+2+3+\cdots+q] + [\,q+1+q+2+\cdots+q+p]$$
$$= 2\left[\frac{q(q+1)}{2}\right] + pq + \frac{p(p+1)}{2}.$$
$qE_{min} = pq + q(q+1) + \frac{p(p+1)}{2}$ Where G is cyclic graph $(p=q)$
$$E_{min} = p + (p+1) + \frac{(p+1)}{2} = \frac{5q+3}{2}.$$

Example 5.3

1. If G is cyclic graph with 3-points

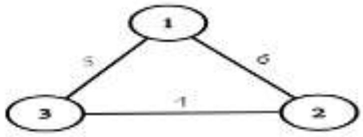

Fig 5.2

then the minimum Edge-magic constant is $E_{min} = 9$.

2. If Gis cyclic graph with 7-points

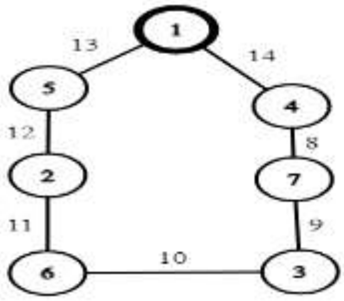

Fig 5.3

then the minimum Edge-magic constant is $E_{min} = 19$.

6. THE GENERAL METHOD TO FIND THE MINIMUM EDGE MAGIC CONSTANT OF CYCLIC GRAPH WHEN P IS ODD

In this sector, we derive the general approach for Minimum Edge-Magic constant of cyclic graph when P is odd.

The vertices of the cyclic graph are labeled with the number $\{1,2,\dots,p\}$ as the following symmetric group way of method

$$\begin{pmatrix} 1 & 2 & 3 & \dots & \frac{\lfloor p \rfloor}{2}+1 & \dots & p \\ \frac{\lfloor p \rfloor}{2} & \frac{\lfloor p \rfloor}{2}+1 & \dots & \dots & 1 & \dots & \frac{\lfloor p \rfloor}{2}-1 \end{pmatrix}$$

Where $\lfloor p \rfloor$ is smallest integer function.The Edges are labeled as the following way. The sum of edges $\frac{\lfloor p \rfloor}{2}+1$ then put $\{1,2,\dots\}$ to the corresponding value.

Edge Labeling

The largest sum of every Edge incident vertices put the Edge label is $p+1$ and continue the next largest to $p+2$ in the manner. The largest sum is $\frac{3p+1}{2}$.

Example 6.1 If G is cyclic graph with 9-points

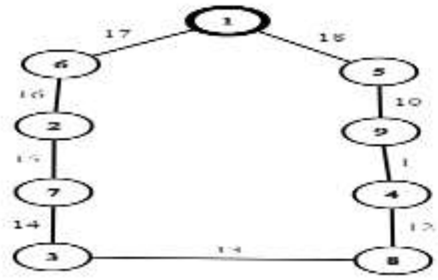

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Fig 6.1

then the

$$E_{min} = \frac{5(9)+3}{2} = 24.$$

The labeling method as follows.

Vertices Label

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\frac{\lfloor p \rfloor}{2} = \frac{\lfloor 9 \rfloor}{2} = 4$$

The value

$\frac{\lfloor p \rfloor}{2} + 1 = 5.$

The largest sum of vertex label of the edge is 14.

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Intro increase
What Gallian. J. A,, Gallon I.S, Gray, I. D, Lo S,
Mac Dougall, J. A. Miller, M. Slamin, and Wallis, W.D,
Marimuthu G and Balakrishnan M,
What the above work???
Namachivayam T, Elumalai M, Sugumaran A, u r previous works
What you done
How it differs form above papers