International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 6s,2025
https://www.theaspd.com/ijes.php

Investigating M-Closed Functions and Homeomorphisms in Fuzzy Hypersoft Topological Structures

V. Subash [1] and M. Angayarkanni [2]
[1]Mathematics Education, M.Shanthi College of Education, CMS Nagar, Ernapuram, Namakkal- 637 003
[2]Department of Mathematics, Kandaswamy Kandar's College, P-Velur, Tamil Nadu-638 182
Corresponding Author: V. Subash
online2020av@gmail.com, subash.maths@gmail.com, angayarkanni66@rediffmail.com

Abstract
The objective of this paper is to introduce and investigate fuzzy hypersoft M-closed maps within the framework of fuzzy hypersoft topological spaces. The study further explores fundamental properties of fuzzy hypersoft M-closed maps, supported by illustrative examples. In addition, the concept is extended to define fuzzy hypersoft M-homeomorphisms and M C-homeomorphisms, along with an analysis of their related characteristics.
Keywords: *fuzzy hypersoft M closed maps, fuzzy hypersoft M homeomorphism, fuzzy hypersoft M C-homeomorphism.*
AMS(2000) Subject classification: *03E72, 54A10, 54A40, 54C05, 54C10.*

INTRODUCTION

Real-world decision-making problems in fields such as medical diagnosis, engineering, economics, management, computer science, artificial intelligence, social sciences, environmental science, and sociology often involve uncertain and imprecise data. Traditional mathematical methods are inadequate for addressing such problems due to their inability to handle data imprecision. To manage uncertainty, Zadeh [11] introduced the concept of fuzzy sets in 1965, where each element is associated with a membership value ranging between 0 and 1. A fuzzy set allows elements of the universe to belong to it with varying degrees, and these degrees are referred to as membership values.

Building on this, Chang [6] developed the notion of fuzzy topological spaces by incorporating a topological structure into fuzzy sets. Later, in 1999, Molodtsov [7] proposed soft set theory as a new mathematical approach to address uncertainty. A soft set is a parameterized family of subsets, where each parameter represents a property, attribute, or characteristic of the objects under consideration. Soft set theory has found numerous applications in areas such as decision-making, optimization, forecasting, and data analysis. Furthermore, Shabir and Naz [8] extended this framework by introducing the concept of soft topological spaces.Smarandache [9] extended the concept of a soft set to a hypersoft set and subsequently to a plithogenic set by replacing the traditional single-argument function with a multi-argument function defined on the Cartesian product of a universe with a distinct set of attributes. This modification allows for a more comprehensive and nuanced representation of information, especially when dealing with complex decision-making problems that involve multiple and diverse attributes. The hypersoft set framework offers greater flexibility compared to the classical soft set, making it more suitable for real-world applications where various criteria need to be considered simultaneously.

Building on this foundation, Abbas et al. [1] formalized the basic operations on hypersoft sets and introduced the notion of a hypersoft point within the universe of discourse. These foundational developments set the stage for further topological exploration in the hypersoft context.

Ajay and Charisma [3] advanced the theory by introducing fuzzy hypersoft topology, intuitionistic hypersoft topology, and neutrosophic hypersoft topology, each extending the traditional topological structures into the hypersoft framework. Among these, neutrosophic hypersoft topology serves as the most generalized form, encompassing and extending both intuitionistic and fuzzy hypersoft topologies.

Further contributions were made by Ajay et al. [4], who defined fuzzy hypersoft semi-open sets and demonstrated their applicability in multi-attribute group decision-making scenarios. This work highlighted the practical relevance of hypersoft topological structures in real-life decision-making

processes.
In parallel, Aras and Bayramov [5] introduced the notion of neutrosophic soft continuity within the setting of neutrosophic soft topological spaces, thereby expanding the analytical tools available for neutrosophic-based decision analysis.
Additionally, Ahsan et al. [2] conducted a detailed theoretical and analytical study focusing on the fundamental framework of composite mappings in the context of fuzzy hypersoft classes, further enriching the mathematical structure and potential applications of hypersoft set theory.
In this paper, we introduce the concept of fuzzy hypersoft M-closed maps within the framework of fuzzy hypersoft topological spaces and examine several of their fundamental properties, supported by illustrative examples. Furthermore, we define and explore the characteristics and properties of fuzzy hypersoft M-homeomorphisms and fuzzy hypersoft $M\ C$-homeomorphisms.

Preliminaries

Definition 2.1 *[11] Let $\mathfrak{Z}$ be an initial universe. A function λ from $\mathfrak{Z}$ into the unit interval I is called a fuzzy set in $\mathfrak{Z}$. For every $\mathfrak{z} \in \mathfrak{Z}, \lambda(\mathfrak{z}) \in I$ is called the grade of membership of $\mathfrak{z}$ in λ. Some authors say that λ is a fuzzy subset of $\mathfrak{Z}$ instead of saying that λ is a fuzzy set in $\mathfrak{Z}$. The class of all fuzzy sets from $\mathfrak{Z}$ into the closed unit interval I will be denoted by $I^{\mathfrak{Z}}$.*

Definition 2.2 *[7] Let $\mathfrak{Z}$ be an initial universe, $\mathfrak{R}$ be a set of parameters and $\mathcal{P}(\mathfrak{Z})$ be the power set of $\mathfrak{Z}$. A pair $(\tilde{I}, \mathfrak{R})$ is called the a soft set over $\mathfrak{Z}$ where $\tilde{I}$ is a mapping $\tilde{I}: \mathfrak{R} \to \mathcal{P}(\mathfrak{R})$. In other words, the soft set is a parametrized family of subsets of the set $\mathfrak{R}$.*

Definition 2.3 *[9] Let $\mathfrak{Z}$ be an initial universe and $\mathcal{P}(\mathfrak{Z})$ be the power set of $\mathfrak{Z}$. Consider $\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3, \ldots, \mathfrak{r}_n$ for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_n$ with $\mathfrak{R}_i \cap \mathfrak{R}_j = \emptyset$, for $i \neq j$ and $i, j \in \{1,2,\ldots,n\}$. Then the pair $(\tilde{I}, \mathfrak{R}_1 \times \mathfrak{R}_2 \times \ldots \times \mathfrak{R}_n)$ where $\tilde{I}: \mathfrak{R}_1 \times \mathfrak{R}_2 \times \ldots \times \mathfrak{R}_n \to \mathcal{P}(\mathfrak{Z})$ is called a hypersoft set over $\mathfrak{Z}$.*

Definition 2.4 *[1] Let $\mathfrak{Z}$ be an initial universal set and $\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_n$ be pairwise disjoint sets of parameters. Let $\mathcal{P}(\mathfrak{Z})$ be the set of all fuzzy sets of $\mathfrak{Z}$. Let $\mathfrak{S}_i$ be the nonempty subset of the pair $\mathfrak{R}_i$ for each $i = 1,2,\ldots,n$. A fuzzy hypersoft set (briefly, $FHSs$) over $\mathfrak{Z}$ is defined as the pair $(\tilde{I}, \mathfrak{S}_1 \times \mathfrak{S}_2 \times \ldots \times \mathfrak{S}_n)$ where $\tilde{I}: \mathfrak{S}_1 \times \mathfrak{S}_2 \times \ldots \times \mathfrak{S}_n \to \mathcal{P}(\mathfrak{Z})$ and $\tilde{I}(\mathfrak{S}_1 \times \mathfrak{S}_2 \times \ldots \times \mathfrak{S}_n) = \{(\mathfrak{r}, \langle \mathfrak{z}, \mu_{\tilde{I}(\mathfrak{r})}(\mathfrak{z})\rangle : \mathfrak{z} \in \mathfrak{Z}) : \mathfrak{r} \in \mathfrak{s}_1 \times \mathfrak{s}_2 \times \ldots \times \mathfrak{s}_n \subseteq \mathfrak{R}_1 \times \mathfrak{R}_2 \times \ldots \times \mathfrak{R}_n\}$ where $\mu_{\tilde{I}(\mathfrak{r})}(\mathfrak{z})$ is the membership value such that $\mu_{\tilde{I}(\mathfrak{r})}(\mathfrak{z}) \in [0,1]$.*

Definition 2.5 *[1] Let $\mathfrak{M}$ be an universal set and $(\tilde{I}, \Delta_1)$ and $(\tilde{J}, \Delta_2)$ be two $FHSs$'s over $\mathfrak{M}$. Then $(\tilde{I}, \Delta_1)$ is the fuzzy hypersoft subset of $(\tilde{J}, \Delta_2)$ if $\mu_{\tilde{I}(\mathfrak{r})}(\mathfrak{z}) \leq \mu_{\tilde{J}(\mathfrak{r})}(\mathfrak{z})$.*
It is denoted by $(\tilde{I}, \Delta_1) \subseteq (\tilde{J}, \Delta_2)$.

Definition 2.6 *[1] Let $\mathfrak{Z}$ be an universal set and $(\tilde{I}, \Delta_1)$ and $(\tilde{J}, \Delta_2)$ be $FHSs$'s over $\mathfrak{Z}$. $(\tilde{I}, \Delta_1)$ is equal to $(\tilde{J}, \Delta_1)$ if $\mu_{\tilde{I}(\ r))}(\mathfrak{z}) = \mu_{\tilde{J}(\ r))}(\mathfrak{z})$.*

Definition 2.7 *[1] A $FHSs$ $(\tilde{I}, \Delta)$ over the universe set $\mathfrak{Z}$ is said to be null fuzzy hypersoft set if $\mu_{\tilde{I}(\mathfrak{r})}(\mathfrak{z}) = 0$, $\forall \mathfrak{r} \in \Delta$ and $\mathfrak{z} \in \mathfrak{Z}$. It is denoted by $\tilde{0}_{(\mathfrak{Z},\mathfrak{R})}$.*
A $FHSs$ $(\tilde{J}, \Delta)$ over the universal set $\mathfrak{Z}$ is said to be absolute fuzzy hypersoft set if $\mu_{\tilde{I}(\mathfrak{r})}(\mathfrak{z}) = 1\ \forall \mathfrak{r} \in \Delta$ and $\mathfrak{z} \in \mathfrak{M}$. It is denoted by $\tilde{1}_{(\mathfrak{Z},\mathfrak{R})}$.
Clearly, $\tilde{0}^c_{(\mathfrak{Z},\mathfrak{R})} = \tilde{1}_{(\mathfrak{Z},\mathfrak{R})}$ and $\tilde{1}^c_{(\mathfrak{Z},\mathfrak{R})} = \tilde{0}_{(\mathfrak{Z},\mathfrak{R})}$.

Definition 2.8 *[1] Let $\mathfrak{Z}$ be an universal set and $(\tilde{I}, \Delta)$ be $FHSs$ over $\mathfrak{Z}$. $(\tilde{I}, \Delta)^C$ is the complement of $(\tilde{I}, \Delta)$ if $\mu^C_{\tilde{I}(\mathfrak{r})}(\mathfrak{z}) = \tilde{1}^c_{(\mathfrak{Z},\mathfrak{R})} - \mu_{\tilde{I}(\mathfrak{r})}(\mathfrak{z})$ where $\forall \mathfrak{r} \in \Delta$ and $\forall \mathfrak{z} \in \mathfrak{Z}$. It is clear that $((\tilde{I}, \Delta)^C)^C = (\tilde{I}, \Delta)$.*

Definition 2.9 *[1] Let $\mathfrak{Z}$ be the universal set and $(\tilde{I}, \Delta_1)$ and $(\tilde{J}, \Delta_2)$ be $FHSs$'s over $\mathfrak{Z}$. Extended union $(\tilde{I}, \Delta_1) \cup (\tilde{J}, \Delta_2)$ is defined as*

$$\mu((\tilde{I}, \Delta_1) \cup (\tilde{J}, \Delta_2)) = \begin{cases} \mu_{\tilde{I}(\mathfrak{r})}(\mathfrak{z}) & \text{if } \ \mathfrak{r} \in \Delta_1 - \Delta_2 \\ \mu_{\tilde{J}(\mathfrak{r})}(\mathfrak{z}) & \text{if } \ \mathfrak{r} \in \Delta_2 - \Delta_1 \\ \max\{\mu_{\tilde{I}(\mathfrak{r})}(\mathfrak{z}), \mu_{\tilde{J}(\mathfrak{r})}(\mathfrak{z})\} & \text{if } \mathfrak{r} \in \Delta_1 \cap \Delta_2 \end{cases}$$

Definition 2.10 *[1, 3] Let $\mathfrak{Z}$ be the universal set and $(\tilde{I}, \Delta_1)$ and $(\tilde{J}, \Delta_2)$ be $FHSs$'s over $\mathfrak{Z}$. Extended intersection $(\tilde{I}, \Delta_1) \cap (\tilde{J}, \Delta_2)$ is defined as*

$$\mu((\tilde{I},\Delta_1)\cap(\tilde{J},\Delta_2))=\begin{cases}\mu_{\tilde{I}(\mathfrak{r})}(\mathfrak{z}) & if\,\mathfrak{r}\in\Delta_1-\Delta_2\\ \mu_{\tilde{J}(\mathfrak{r})}(\mathfrak{z}) & if\,\mathfrak{r}\in\Delta_2-\Delta_1\\ \min\{\mu_{\tilde{I}(\mathfrak{r})}(\mathfrak{z}),\mu_{\tilde{J}(\mathfrak{r})}(\mathfrak{z})\} & if\,\mathfrak{r}\in\Delta_1\cap\Delta_2\end{cases}$$

Definition 2.11 *[3] Let $(\mathfrak{Z},\mathfrak{R})$ be the family of all $FHSs$'s over the universe set $\mathfrak{Z}$ and $\tau\subseteq FHSs(\mathfrak{Z},\mathfrak{R})$. Then τ is said to be a fuzzy hypersoft topology (briefly, $FHSt$) on $\mathfrak{Z}$ if*

1. $\tilde{0}_{(\mathfrak{Z},\mathfrak{R})}$ and $\tilde{1}_{(\mathfrak{Z},\mathfrak{R})}$ belongs to τ
2. The union of any number of $FHSs$'s in τ belongs to τ
3. The intersection of finite number of $FHSs$'s in τ belongs to τ.

Then $(\mathfrak{Z},\mathfrak{R},\tau)$ is called a fuzzy hypersoft toplogical space (briefly, $FHSts$) over $\mathfrak{Z}$. Each member of τ is said to be fuzzy hypersoft open set (briefly, $FHSos$). A $FHSs$ $(\tilde{I},\Delta)$ is called a fuzzy hypersoft closed set (briefly, $FHScs$) if its complement $(\tilde{I},\Delta)^C$ is $FHSos$.

Definition 2.12 *[3] Let $(\mathfrak{Z},\mathfrak{R},\tau)$ be a $FHSts$ over $\mathfrak{Z}$ and $(\tilde{I},\Delta)$ be a $FHSs$ in $\mathfrak{Z}$. Then,*

1. The fuzzy hypersoft interior (briefly, $FHSint$) of $(\tilde{I},\Delta)$ is defined as $FHSint(\tilde{I},\Delta)=\cup\ \{(\tilde{J},\Delta):(\tilde{J},\Delta)\subseteq(\tilde{I},\Delta)$ where $(\tilde{J},\Delta)$ is $FHSos\}$.
2. The fuzzy hypersoft closure (briefly, $FHScl$) of $(\tilde{I},\Delta)$ is defined as $FHScl(\tilde{I},\Delta)=\cap\ \{(\tilde{J},\Delta):(\tilde{J},\Delta)\supseteq(\tilde{I},\Delta)$ where $(\tilde{J},\Delta)$ is $FHScs\}$.

Definition 2.13 *[4] Let $(\mathfrak{Z},\mathfrak{R},\tau)$ be a $FHSts$ over $\mathfrak{Z}$ and $(\tilde{I},\Delta)$ be a $FHSs$ in $\mathfrak{Z}$. Then, $(\tilde{I},\Delta)$ is called the fuzzy hypersoft semiopen set (briefly, $FHSSos$) if $(\tilde{I},\Delta)\subseteq FHScl(int(\tilde{I},\Delta))$.*
A $FHSs$ $(\tilde{I},\Delta)$ is called a fuzzy hypersoft semiclosed set (briefly, $FHSScs$) if its complement $(\tilde{I},\Delta)^c$ is a $FHSSos$.

Definition 2.14 *[2] Let $(\mathfrak{Z},\mathfrak{L})$ and $(\mathfrak{Y},\mathfrak{M})$ be classes of $FHSs$'s over $\mathfrak{Z}$ and $\mathfrak{Y}$ with attributes $\mathfrak{L}$ and $\mathfrak{M}$ respectively. Let $\omega:\mathfrak{Z}\to\mathfrak{Y}$ and $\nu:\mathfrak{L}\to\mathfrak{M}$ be mappings. Then a FHS mappings $\mathfrak{h}=(\omega,\nu):(\mathfrak{Z},\mathfrak{L})\to(\mathfrak{Y},\mathfrak{M})$ is defined as follows, for a $FHSs$ $(\tilde{I},\Delta)_A$ in $(\mathfrak{Z},\mathfrak{L})$, $f(\tilde{I},\Delta)_A$ is a $FHSs$ in $(\mathfrak{Y},\mathfrak{M})$ obtained as follows, for $\beta\in\nu(\mathfrak{L})\subseteq\mathfrak{M}$ and $\mathfrak{y}\in\mathfrak{y}$, $\mathfrak{h}(\tilde{I},\Delta)_A(\beta)(\mathfrak{y})=\cup_{\alpha\in\nu^{-1}(\beta)\cap\ A,s\in\omega^{-1}(\mathfrak{y})}(\alpha)\mu_s\mathfrak{h}(\tilde{I},\Delta)_A$ is called a fuzzy hypersoft image of a $FHSs$ $(\tilde{I},\Delta)$. Hence $((\tilde{I},\Delta)_A,\mathfrak{h}(\tilde{I},\Delta)_A)\in\mathfrak{h}$, where $(\tilde{I},\Delta)_A\subseteq(\mathfrak{Z},\mathfrak{L}),\mathfrak{h}(\tilde{I},\Delta)_A\subseteq(\mathfrak{Y},\mathfrak{M})$.*

Definition 2.15 *[2] If $\mathfrak{h}:(\mathfrak{Z},\mathfrak{L})\to(\mathfrak{Y},\mathfrak{M})$ be a FHS mapping, then FHS class $(\mathfrak{Z},\mathfrak{L})$ is called the domain of $\mathfrak{h}$ and the FHS class $(\tilde{J},\Delta)\in(\mathfrak{Y},\mathfrak{M}):(\tilde{J},\Delta)=\mathfrak{h}(\tilde{I},\Delta))$ for some $(\tilde{I},\Delta)\in(\mathfrak{Z},\mathfrak{L})$ is called the range of $\mathfrak{h}$. The FHS class $(\mathfrak{Y},\mathfrak{M})$ is called co-domain of $\mathfrak{h}$.*

Definition 2.16 *[2] If $\mathfrak{h}:(\mathfrak{Z},\mathfrak{L})\to(\mathfrak{Y},\mathfrak{M})$ be a FHS mapping and $(\tilde{J},\Delta)_B$, a $FHSs$ in $(\mathfrak{Y},\mathfrak{Z})$ where $\omega:\mathfrak{Z}\to\mathfrak{Y}$, $\nu:\mathfrak{L}\to\mathfrak{M}$ and $B\subseteq\mathfrak{M}$. Then $\mathfrak{h}^{-1}(\tilde{J},\Delta)_B$ is a $FHSs$ in $(\mathfrak{Z},\mathfrak{L})$ defined as follows, for $\alpha\in\nu^{-1}(B)\subseteq\mathfrak{L}$ and $\mathfrak{z}\in\mathfrak{Z}$, $\mathfrak{h}^{-1}(\tilde{J},\Delta)_B(\alpha)(\mathfrak{z})=(\nu(\alpha))\mu_p(\mathfrak{z})\mathfrak{h}^{-1}(\tilde{J},\Delta)_B$ is called a FHS inverse image of $(\tilde{J},\Delta)_B$.*

Definition 2.17 *[2] Let $\mathfrak{h}=(\omega,\nu)$ be a FHS mapping of a FHS class $(\mathfrak{Z},\mathfrak{L})$ into a FHS class $(\mathfrak{Y},\mathfrak{M})$. Then*

1. $\mathfrak{h}$ is said to be a one-one (or injection) FHS mapping if for both $\omega:\mathfrak{Z}\to\mathfrak{Y}$ and $\nu:\mathfrak{L}\to\mathfrak{M}$ are one-one.
2. $\mathfrak{h}$ is said to be a onto (or surjection) FHS mapping if for both $\omega:\mathfrak{Z}\to\mathfrak{Y}$ and $\nu:\mathfrak{L}\to\mathfrak{M}$ are onto.

If $\mathfrak{h}$ is both one-one and onto, then $\mathfrak{h}$ is called a one-one correspondance (or bijective) of FHS mapping.

Definition 2.18 *[2] If $\mathfrak{h}=(\omega,\nu):(\mathfrak{Z},\mathfrak{L})\to(\mathfrak{Y},\mathfrak{M})$ and $g=(\mathfrak{m},\mathfrak{y}):(\mathfrak{Y},\mathfrak{M})\to(\mathfrak{X},\mathfrak{N})$ are two FHS mappings, then their composite $\mathfrak{g}\circ\mathfrak{h}$ is a FHS mapping of $(\mathfrak{Z},\mathfrak{L})$ into $(\mathfrak{X},\mathfrak{N})$ such that for every $(\tilde{I},\Delta)_A\in(\mathfrak{Z},\mathfrak{L})$, $(\mathfrak{g}\circ\mathfrak{h})(\tilde{I},\Delta)_A=\mathfrak{g}(\mathfrak{h}(\tilde{I},\Delta)_A)$. For $\beta\in\mathfrak{y}(\mathfrak{M})\subseteq\mathfrak{N}$ and $\mathfrak{x}\in\mathfrak{X}$, it is defined as $\mathfrak{g}(\mathfrak{h}(\tilde{I},\Delta)_A(\beta)(\mathfrak{x})=\cup_{\alpha\in\mathfrak{y}^{-1}(\beta)\cap\ \mathfrak{h}(A),s\in\mathfrak{z}^{-1}(\mathfrak{x})}(\alpha)\mu_s$.*

Definition 2.19 *[2] Let $\mathfrak{h}=(\omega,\nu)$ be a FHS mapping where $\omega:\mathfrak{Z}\to\mathfrak{Z}$ and $\nu:\mathfrak{L}\to\mathfrak{L}$. Then $\mathfrak{h}$ is said to be a FHS identity mapping if for both ω and ν are identity mappings.*

Definition 2.20 *[2] A one-one onto FHS mapping $\mathfrak{h} = (\omega, \nu): (\mathfrak{Z}, \mathfrak{L}) \to (\mathfrak{Y}, \mathfrak{M})$ is called FHS invertable mapping. Its FHS inverse mapping is denoted by $\mathfrak{h}^{-1} = (\omega^{-1}, \nu^{-1}): (\mathfrak{Z}, \mathfrak{L}) \to (\mathfrak{Y}, \mathfrak{M})$.*

Definition 2.21 *[10] Let $(\mathfrak{Z}, \mathfrak{R}, \tau)$ be a $FHSts$ over $\mathfrak{Z}$ and $(\tilde{I}, \Delta)$ be a $FHSs$ on $\mathfrak{Z}$. Then the fuzzy hypersoft*

1. δ-interior (briefly, $FHSint$) of $(\tilde{I}, \Delta)$ is defined by

$FHS\delta int(\tilde{I}, \Delta) = \cup \ \{(\tilde{J}, \Delta): (\tilde{J}, \Delta) \subseteq (\tilde{I}, \Delta)$ and $(\tilde{J}, \Delta)$ is a $FHSros$ in $\mathfrak{Z}\}$

2. δ-closure (briefly, $FHScl$) of $(\tilde{I}, \Delta)$ is defined by

$FHS\delta cl(\tilde{I}, \Delta) = \cap \ \{(\tilde{J}, \Delta): (\tilde{J}, \Delta) \supseteq (\tilde{I}, \Delta)$ and $(\tilde{J}, \Delta)$ is a $FHSrcs$ in $\mathfrak{Z}\}$

Definition 2.22 *[10] Let $(\mathfrak{Z}, \mathfrak{R}, \tau)$ be a $FHSts$ over $\mathfrak{Z}$. A $FHSs$ $(\tilde{I}, \Delta)$ is said to be a fuzzy hypersoft*

1. semi-regular if $(\tilde{I}, \Delta)$ is both $FHSSos$ and $FHSScs$.
2. pre open set (briefly, $FHS\mathcal{P}os$) if $(\tilde{I}, \Delta) \subseteq FHSint(FHScl(\tilde{I}, \Delta)$
3. δ-open set (briefly, $FHS\delta os$) if $(\tilde{I}, \Delta) = FHS\delta int(\tilde{I}, \Delta)$
4. δ-pre open set (briefly, $FHS\delta\mathcal{P}os$) if $(\tilde{I}, \Delta) \subseteq FHSint(FHS\delta cl(\tilde{I}, \Delta))$
5. δ-semi open set (briefly, $FHS\delta\mathcal{S}os$) if $(\tilde{I}, \Delta) \subseteq FHScl(FHS\delta int(\tilde{I}, \Delta))$
6. e-open set (briefly, $FHSeos$) if $(\tilde{I}, \Delta) \subseteq FHScl(FHS\delta int(\tilde{I}, \Delta)) \cup FHSint(FHS\delta cl(\tilde{I}, \Delta))$.

The complement of $FHS\delta os$ (resp. $FHS\mathcal{P}os$, $FHS\delta\mathcal{P}os$ $FHS\delta\mathcal{S}os$ & $FHSeos$) is called a $FHS\delta$ (resp. FHS pre, $FHS\delta$ pre $FHS\delta$ semi & $FHSe$) closed set (briefly, $FHS\delta cs$ (resp. $FHS\mathcal{P}cs$, $FHS\delta\mathcal{P}cs$ $FHS\delta\mathcal{S}cs$ & $FHSecs$)) in $\mathfrak{Z}$.
The family of all $FHS\delta os$ (resp. $FHS\delta cs$, $FHSros$, $FHSrcs$, $FHS\mathcal{P}os$, $FHS\mathcal{P}cs$ $FHS\delta\mathcal{P}os$, $FHS\delta\mathcal{P}cs$, $FHS\delta\mathcal{S}os$ $FHS\delta\mathcal{S}cs$, $FHSeos$ & $FHSecs$) of Y is denoted by $FHS\delta OS(\mathfrak{Z})$ (resp. $FHS\delta CS(\mathfrak{Z})$, $FHSrOS(\mathfrak{Z})$, $FHSrOS(\mathfrak{Z})$, $FHS\ \mathcal{P}OS(\mathfrak{Z})$, $FHS\mathcal{P}CS(\mathfrak{Z})$, $FHS\delta\mathcal{P}OS(\mathfrak{Z})$, $FHS\delta\mathcal{P}CS(\mathfrak{Z})$, $FHS\delta\mathcal{S}OS(\mathfrak{Z})$, $FHS\delta\mathcal{S}CS(\mathfrak{Z})$, $FHSeOS(\mathfrak{Z})$& $FHSeCS(\mathfrak{Z})$).

Fuzzy hypersoft *M*-closed mapping

In this section, we introduce fuzzy hypersoft M closed maps and study their characteristics.

Definition 3.1 *Let $(\mathfrak{Z}, \mathfrak{Z}, \tau)$ be a $FHSts$ over $\mathfrak{Z}$ and $(\tilde{I}, \Delta)$ be a $FHSs$ on $\mathfrak{Z}$. Then the fuzzy hypersoft*

1. θ-interior (briefly, $FHS\theta int$) of $(\tilde{I}, \Delta)$ is defined by

$FHS\theta int(\tilde{I}, \Delta) = \cup \ \{FHSint(\tilde{J}, \Delta): (\tilde{J}, \Delta) \subseteq (\tilde{I}, \Delta)$ and $(\tilde{J}, \Delta)$ is a $FHScs$ in $\mathfrak{Z}\}$

2. θ-closure (briefly, $FHS\theta cl$) of $(\tilde{I}, \Delta)$ is defined by

$FHS\theta cl(\tilde{I}, \Delta) = \cap \ \{FHScl(\tilde{J}, \Delta): (\tilde{I}, \Delta) \subseteq (\tilde{J}, \Delta)$ and $(\tilde{J}, \Delta)$ is a $FHSos$ in $\mathfrak{Z}\}$

Definition 3.2 *Let $(\mathfrak{Z}, \mathfrak{Z}, \tau)$ be a $FHSts$ over $\mathfrak{Z}$. A $FHSs$ $(\tilde{I}, \Delta)$ is said to be a fuzzy hypersoft*

1. θ-open set (briefly, $FHS\theta os$) if $(\tilde{I}, \Delta) = FHS\theta int(\tilde{I}, \Delta)$
2. θ-pre open set (briefly, $FHS\theta\mathcal{P}os$) if $(\tilde{I}, \Delta) \subseteq FHSint(FHS\theta cl(\tilde{I}, \Delta))$
3. θ-semi open set (briefly, $FHS\theta\mathcal{S}os$) if $(\tilde{I}, \Delta) \subseteq FHScl(FHS\theta int(\tilde{I}, \Delta))$
4. M- open set(briefly, $FHSMos$) if $(\tilde{I}, \Delta) \subseteq FHScl(FHS\theta int(\tilde{I}, \Delta)) \cup FHSint(FHS\delta cl(\tilde{I}, \Delta))$

The complement of $FHS\theta os$ (resp. $FHS\theta\mathcal{P}os$, $FHS\theta\mathcal{S}os$ & $FHSMos$) is called a $FHS\theta$ (resp. $FHS\theta$ pre , $FHS\theta$ semi & $FHSM$) closed set (briefly, $FHS\theta cs$ (resp. $FHS\theta\mathcal{P}cs$, $FHS\theta\mathcal{S}cs$ & $FHSMcs$)) in $\mathfrak{Z}$.
The family of all $FHS\theta os$ (resp. $FHS\theta cs$, $FHS\theta\mathcal{P}os$, $FHS\theta\mathcal{P}cs$, $FHS\theta\mathcal{S}os$, $FHS\theta\mathcal{S}cs$, $FHSMos$ &$FHSMcs$) of $\mathfrak{Z}$ is denoted by $FHS\theta OS(\mathfrak{Z})$ (resp. $FHS\theta CS(\mathfrak{Z})$, $FHS\theta\mathcal{P}OS(\mathfrak{Z})$, $FHS\theta\mathcal{P}CS(\mathfrak{Z})$, $FHS\theta\mathcal{S}OS(\mathfrak{Z})$, $FHS\theta\mathcal{S}CS(\mathfrak{Z})$, $FHSMOS(\mathfrak{Z})$ & $FHSMCS(\mathfrak{Z})$).

Definition 3.3 *Let $(\mathfrak{Z}, \mathfrak{R}, \tau)$ be a $FHSts$ over $\mathfrak{Z}$ and $(\tilde{I}, \Delta)$ be a $FHSs$ on $\mathfrak{Z}$. Then the fuzzy hypersoft*

1. M-interior (briefly, $FHSMint$) of $(\tilde{I}, \Delta)$ is defined by

$FHSMint(\tilde{I}, \Delta)$=$\cup \ \{(\tilde{J}, \Delta): (\tilde{J}, \Delta) \subseteq (\tilde{I}, \Delta)$ & $(\tilde{J}, \Delta)$ is a $FHSMos$ in $\mathfrak{Z}\}$

2. M-closure (briefly, $FHSMcl$) of $(\tilde{I}, \Delta)$ is defined by

$FHSMcl(\tilde{I}, \Delta)$=$\cap \ \{(\tilde{J}, \Delta): (\tilde{I}, \Delta) \subseteq (\tilde{J}, \Delta)$ & $(\tilde{J}, \Delta)$ is a $FHSMcs$ in $\mathfrak{Z}\}$

Definition 3.4 *Consider any two FHSts* $(\mathfrak{Z}, \mathfrak{L}, \tau)$ *and* $(\mathfrak{Y}, \mathfrak{M}, \sigma)$. *A map* $\mathfrak{h}: (\mathfrak{Z}, \mathfrak{L}, \tau) \to (\mathfrak{Y}, \mathfrak{M}, \sigma)$ *is said to be* FHS1. continuous (briefly, $FHSCts$) if the inverse image of each $FHSos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ is a $FHSos$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$.
2. θ continuous (briefly, $FHS\theta Cts$) if the inverse image of each $FHSos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ is a $FHS\theta os$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$.
3. δpre continuous (briefly, $FHS\delta\mathcal{P}Cts$) if the inverse image of each $FHSos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ is a $FHS\delta\mathcal{P}os$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$.
4. $\delta semi$ continuous (briefly, $FHS\delta SCts$) if the inverse image of each $FHSos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ is a $FHS\delta Sos$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$.
5. θS continuous (briefly, $FHS\theta SCts$) if the inverse image of each $FHSos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ is a $FHS\theta Sos$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$.
6. δ continuous (briefly, $FHS\delta Cts$) if the inverse image of each $FHSos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ is a $FHS\delta os$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$.
7. M continuous (briefly, $FHSMCts$) if the inverse image of each $FHSos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ is a $FHSMos$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$.
8. e continuous (briefly, $FHSeCts$) if the inverse image of each $FHSos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ is a $FHSeos$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$.

Definition 3.5 *A mapping* $\mathfrak{h}: (\mathfrak{Z}, \mathfrak{L}, \tau) \to (\mathfrak{Y}, \mathfrak{M}, \sigma)$ *is called* FHS
1. open (in short, $FHSO$) mapping if the image of every $FHSos$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHSos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
2. δ-open (in short, $FHS\delta O$) mapping if the image of every $FHSos$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\delta os$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
3. δ-semi open (in short, $FHS\delta SO$) mapping if the image of every $FHSos$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\delta Sos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
4. δ-pre open (in short, $FHS\theta\mathcal{P}O$) mapping if the image of every $FHSos$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\delta\mathcal{P}os$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
5. θ-open (in short, $FHS\theta O$) mapping if the image of every $FHSos$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\theta os$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
6. θ-semi open (in short, $FHS\theta SO$) mapping if the image of every $FHSos$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\theta Sos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
7. θ-pre open (in short, $FHS\theta\mathcal{P}O$) mapping if the image of every $FHSos$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\theta\mathcal{P}os$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
8. M-open (in short, $FHSMO$) mapping if the image of every $FHSos$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHSMos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
9. e-open (in short, $FHSeO$) mapping if the image of every $FHSos$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHSeos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.

Definition 3.6 *A mapping* $\mathfrak{h}: (\mathfrak{Z}, \mathfrak{L}, \tau) \to (\mathfrak{Y}, \mathfrak{M}, \sigma)$ *is called* FHS
1. close (in short, $FHSC$) mapping if the image of every $FHScs$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHScs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
2. δ-closed (in short, $FHS\delta C$) mapping if the image of every $FHScs$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\delta cs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
3. δ-semi closed (in short, $FHS\delta SC$) mapping if the image of every $FHScs$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\delta Scs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
4. δ-pre closed (in short, $FHS\theta\mathcal{P}C$) mapping if the image of every $FHScs$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\delta\mathcal{P}cs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
5. θ-closed (in short, $FHS\theta C$) mapping if the image of every $FHScs$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\theta cs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
6. θ-semi closed (in short, $FHS\theta SC$) mapping if the image of every $FHScs$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\theta Scs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
7. θ-pre closed (in short, $FHS\theta\mathcal{P}C$) mapping if the image of every $FHScs$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHS\theta\mathcal{P}cs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.

8. M-closed (in short, $FHSMC$) mapping if the image of every $FHScs$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHSMcs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.
9. e-closed (in short, $FHSeC$) mapping if the image of every $FHScs$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is $FHSecs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.

Example 3.1 *Let $\mathfrak{Z} = \{\mathfrak{z}_1, \mathfrak{z}_2\}$ and $\mathfrak{Y} = \{\mathfrak{y}_1, \mathfrak{y}_2\}$ be the FHS initial universes and the attributes be $\mathfrak{L} = \mathfrak{R}_1 \times \mathfrak{R}_2$ and $\mathfrak{M} = \mathfrak{R}_{1'} \times \mathfrak{R}_{2'}$ respectively. The attributes are given as:*

$\mathfrak{R}_1 = \{a_1, a_2, a_3\}, \mathfrak{R}_2 = \{b_1, b_2\}$

$\mathfrak{R}_{1'} = \{c_1, c_2, c_3\}, \mathfrak{R}_{2'} = \{d_1, d_2\}$.

Let $(\mathfrak{Z}, \mathfrak{L})$ and $(\mathfrak{Y}, \mathfrak{M})$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{I}_1, \Delta_1)$ over the universe $\mathfrak{M}$ be

$$(\tilde{I}_1\Delta_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{z}_1}{0.8}, \frac{\mathfrak{z}_2}{0.7}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{z}_1}{0.7}, \frac{\mathfrak{z}_2}{0.5}\}\rangle \end{Bmatrix}$$

$$(\tilde{I}_2, \Delta_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{z}_1}{0.2}, \frac{\mathfrak{z}_2}{0.3}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{z}_1}{0.3}, \frac{\mathfrak{z}_2}{0.5}\}\rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{Z},\mathfrak{R})}, \tilde{1}_{(\mathfrak{Z},\mathfrak{R})}, (\tilde{I}_1, \Delta_1), (\tilde{I}_2, \Delta_1)\}$ is $FHSts$.

Let the $FHSs's$ $(\tilde{J}_1, \Delta_1)$ and $(\tilde{J}_2, \Delta_2)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1, \Delta_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.8}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.7}\}\rangle \end{Bmatrix}$$

$$(\tilde{J}_2, \Delta_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.3}, \frac{\mathfrak{y}_2}{0.2}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.3}\}\rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{Y},\mathfrak{R})}, \tilde{1}_{(\mathfrak{Y},\mathfrak{R})}, (\tilde{J}_1, \Delta_1), (\tilde{J}_2, \Delta_1)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu): (\mathfrak{Z}, \mathfrak{L}) \to (\mathfrak{Y}, \mathfrak{M})$ be a FHS mapping as follows:

$\omega(\mathfrak{z}_1) = \mathfrak{y}_2, \omega(\mathfrak{z}_2) = \mathfrak{y}_1,$

$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2),\ \nu(a_1, b_2) = (c_2, d_2)$

$$\mathfrak{h}(\tilde{I}_1, \Delta_1)^c = (\tilde{J}_1, \Delta_1)^c, \mathfrak{h}(\tilde{I}_2, \Delta_1)^c = (\tilde{J}_2, \Delta_1)^c,$$

$\therefore \mathfrak{h} = (\omega, \nu): (\mathfrak{Z}, \mathfrak{L}) \to (\mathfrak{Y}, \mathfrak{M})$ is $FHSMC$ because $(\tilde{J}_1, \Delta_1)^c$ and $(\tilde{J}_2, \Delta_1)^c$ are $FHSMcs$'s. .

Proposition 3.1 *The statements hold but the converse is not.*

1. Each $FHS\theta C$ is a $FHS\theta\mathcal{S}C$.
2. Each $FHS\theta C$ is a $FHSC$.
3. Each $FHS\theta\mathcal{S}C$ is a $FHSMC$.
4. Each $FHS\delta C$ is a $FHSC$.
5. Each $FHS\delta C$ is a $FHS\delta\mathcal{P}C$.
6. Each $FHS\delta C$ is a $FHS\delta\mathcal{S}C$.
7. Each $FHS\delta\mathcal{P}C$ is a $FHSMC$.
8. Each $FHS\delta\mathcal{S}C$ is a $FHSeC$.
9. Each $FHSMC$ is a $FHSeC$.

Proof.

Consider the map $\mathfrak{h}: (\mathfrak{Z}, \mathfrak{L}, \tau) \to (\mathfrak{Y}, \mathfrak{M}, \sigma)$

1. Let $(\tilde{I}, \Delta)$ be a $FHSos$ in $\mathfrak{Z}$. As $\mathfrak{h}$ is $FHS\theta C$, $\mathfrak{h}(\tilde{I}, \Delta)$ is $FHS\theta cs$ in $\mathfrak{Y}$. Since all $FHS\theta cs$ are $FHS\theta\mathcal{S}cs$, $\mathfrak{h}(\tilde{I}, \Delta)$ is $FHS\theta\mathcal{S}cs$ in $\mathfrak{Y}$. Thus $\mathfrak{h}$ is a $FHS\theta\mathcal{S}C$.
2. Let $(\tilde{I}, \Delta)$ be a $FHScs$ in $\mathfrak{Z}$. As $\mathfrak{h}$ is $FHS\theta C$, $\mathfrak{h}(\tilde{I}, \Delta)$ is $FHS\theta cs$ in $\mathfrak{Y}$. Since all $FHS\theta cs$ are

$FHScs$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHScs$ in $\mathfrak{Y}$. Thus $\mathfrak{h}$ is a $FHSC$.

3. Let $(\tilde{I},\Delta)$ be a $FHScs$ in $\mathfrak{Z}$. As $\mathfrak{h}$ is $FHS\theta\mathcal{S}C$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHS\theta\mathcal{S}cs$ in $\mathfrak{Y}$. Since all $FHS\theta\mathcal{S}os$ are $FHSMcs$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHSMcs$ in $\mathfrak{Y}$. Thus $\mathfrak{h}$ is a $FHSMC$.

4. Let $(\tilde{I},\Delta)$ be a $FHScs$ in $\mathfrak{Z}$. As $\mathfrak{h}$ is $FHS\delta C$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHS\delta cs$ in $\mathfrak{Y}$. Since all $FHS\delta cs$ are $FHScs$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHScs$ in $\mathfrak{Y}$. Thus $\mathfrak{h}$ is a $FHSC$.

5. Let $(\tilde{I},\Delta)$ be a $FHScs$ in $\mathfrak{Z}$. As $\mathfrak{h}$ is $FHS\delta C$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHS\delta cs$ in $\mathfrak{Y}$. Since all $FHS\delta cs$ are $FHS\delta\mathcal{P}cs$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHS\delta\mathcal{P}cs$ in $\mathfrak{Y}$. Thus $\mathfrak{h}$ is a $FHS\delta\mathcal{P}C$.

6. Let $(\tilde{I},\Delta)$ be a $FHScs$ in $\mathfrak{Z}$. As $\mathfrak{h}$ is $FHS\delta C$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHS\delta cs$ in $\mathfrak{Y}$. Since all $FHS\delta cs$ are $FHS\delta\mathcal{S}cs$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHS\delta\mathcal{S}cs$ in $\mathfrak{Y}$. Thus $\mathfrak{h}$ is a $FHS\delta\mathcal{S}C$.

7. Let $(\tilde{I},\Delta)$ be a $FHScs$ in $\mathfrak{Z}$. As $\mathfrak{h}$ is $FHS\delta\mathcal{P}C$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHS\delta\mathcal{P}cs$ in $\mathfrak{Y}$. Since all $FHS\delta\mathcal{P}cs$ are $FHSMcs$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHSMcs$ in $\mathfrak{Y}$. Thus $\mathfrak{h}$ is a $FHSMC$.

8. Let $(\tilde{I},\Delta)$ be a $FHScs$ in $\mathfrak{Z}$. As $\mathfrak{h}$ is $FHS\delta\mathcal{S}C$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHS\delta\mathcal{S}cs$ in $\mathfrak{Y}$. Since all $FHS\delta\mathcal{S}cs$ are $FHSecs$, $\mathfrak{h}\tilde{I},\Delta)$ is $FHSecs$ in $\mathfrak{Y}$. Thus $\mathfrak{h}$ is a $FHSeC$.

9. Let $(\tilde{I},\Delta)$ be a $FHScs$ in $\mathfrak{Z}$. As $\mathfrak{h}$ is $FHSMC$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHSMcs$ in $\mathfrak{Y}$. Since all $FHSMcs$ are $FHSecs$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHSecs$ in $\mathfrak{Y}$. Thus $\mathfrak{h}$ is a $FHSeC$.

width 0.22 true cm height 0.22 true cm depth 0pt

Example 3.2 *Let $\mathfrak{Z} = \{\mathfrak{z}_1,\mathfrak{z}_2\}$ and $\mathfrak{Y} = \{\mathfrak{y}_1,\mathfrak{y}_2\}$ be the FHS initial universes and the attributes be $\mathfrak{L} = \mathfrak{R}_1 \times \mathfrak{R}_2$ and $\mathfrak{M} = \mathfrak{R}_{1'} \times \mathfrak{R}_{2'}$ respectively. The attributes are given as:*

$\mathfrak{R}_1 = \{a_1,a_2,a_3\}, \mathfrak{R}_2 = \{b_1,b_2\}$

$\mathfrak{R}_{1'} = \{c_1,c_2,c_3\}, \mathfrak{R}_{2'} = \{d_1,d_2\}$.

Let $(\mathfrak{Z},\mathfrak{L})$ and $(\mathfrak{Y},\mathfrak{M})$ be the classes of FHS sets. Let the $FHSs$ $(\tilde{I}_1,\Delta_3)$ over the universe $\mathfrak{Z}$ be defined as

$$(\tilde{I}_1,\Delta_3) = \begin{Bmatrix} \langle (a_1,b_1), \{\frac{\mathfrak{z}_1}{0.7},\frac{\mathfrak{z}_2}{0.8}\}\rangle, \\ \langle (a_2,b_1), \{\frac{\mathfrak{z}_1}{0.5},\frac{\mathfrak{z}_2}{0.7}\}\rangle, \\ \langle (a_1,b_1), \{\frac{\mathfrak{z}_1}{0.6},\frac{\mathfrak{z}_2}{0.5}\}\rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{Z},\mathfrak{R})}, \tilde{1}_{(\mathfrak{Z},\mathfrak{R})}, (\tilde{I}_1,\Delta_3)\}$ is $FHSts$.

Let the $FHSs$'s $(\tilde{J}_1,\Delta_1), (\tilde{J}_2,\Delta_2), (\tilde{J}_3,\Delta_2), (\tilde{J}_4,\Delta_3), (\tilde{J}_5,\Delta_3), (\tilde{J}_6,\Delta_3)$ and $(\tilde{J}_7,\Delta_3)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1,\Delta_1) = \begin{Bmatrix} \langle (c_2,d_1), \{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle, \\ \langle (c_1,d_2), \{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle \end{Bmatrix}$$

$$(\tilde{J}_2,\Delta_2) = \begin{Bmatrix} \langle (c_2,d_1), \{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle, \\ \langle (c_2,d_2), \{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{J}_3,\Delta_2) = \begin{Bmatrix} \langle (c_2,d_1), \{\frac{1}{0.8},\frac{\mathfrak{y}_2}{0.7}\}\rangle, \\ \langle (c_2,d_2), \{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.6}\}\rangle \end{Bmatrix}$$

$$(\tilde{J}_4,\Delta_3) = \begin{Bmatrix} \langle (c_2,d_1), \{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle, \\ \langle (c_1,d_2), \{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle, \\ \langle (c_2,d_2), \{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{J}_5,\Delta_3) = \begin{Bmatrix} \langle (c_2,d_1), \{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle, \\ \langle (c_1,d_2), \{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle, \\ \langle (c_2,d_2), \{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{J}_6,\Delta_3)=\begin{Bmatrix}\langle (c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.7}\}\rangle,\\ \langle (c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle (c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.6}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_7,\Delta_3)=\begin{Bmatrix}\langle (c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle (c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle (c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.6}\}\rangle\end{Bmatrix}$$

$\sigma=\{\tilde{0}_{(\mathfrak{Y},\mathfrak{R})},\tilde{1}_{(\mathfrak{Y},\mathfrak{R})},(\tilde{J}_1,\Delta_1),(\tilde{J}_2,\Delta_2),(\tilde{J}_3,\Delta_2),(\tilde{J}_4,\Delta_3),(\tilde{J}_5,\Delta_3),(\tilde{J}_6,\Delta_3),(\tilde{J}_7,\Delta_3)\}$ is $FHSts$.
Let $\mathfrak{h}=(\omega,\nu):(\mathfrak{Z},\mathfrak{L})\to(\mathfrak{Z},\mathfrak{L})$ be a FHS mapping as follows:
$\omega(\mathfrak{z}_1)=\mathfrak{y}_2,\omega(\mathfrak{z}_2)=\mathfrak{y}_1,$
$\nu(a_1,b_1)=(c_2,d_1),\nu(a_2,b_1)=(c_1,d_2),\ \nu(a_1,b_2)=(c_2,d_2)$
$\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_6,\Delta_3)^c$
$(\tilde{I}_1,\Delta_3)^c$ is $FHScs$ in $\mathfrak{Z}$ and $\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_6,\Delta_3)^c$ is $FHS\theta\mathcal{S}cs$ in $\mathfrak{Y}$.
$\therefore$ $\mathfrak{h}$ is $FHS\theta SC$ mapping but not $FHS\theta C$ mapping because $(\tilde{I}_1,\Delta_3)^c$ is $FHScs$ in $\mathfrak{Z}$ but $\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_6,\Delta_3)^c$ is not $FHS\theta cs$ in $\mathfrak{Y}$.

Example 3.3 *Let $\mathfrak{Z}=\{\mathfrak{z}_1,\mathfrak{z}_2\}$ and $\mathfrak{Y}=\{\mathfrak{y}_1,\mathfrak{y}_2\}$ be the FHS initial universes and the attributes be $\mathfrak{L}=\mathfrak{R}_1\times\mathfrak{R}_2$ and $\mathfrak{M}=\mathfrak{R}_{1'}\times\mathfrak{R}_{2'}$, respectively. The attributes are given as:*
$\mathfrak{R}_1=\{a_1,a_2,a_3\},\mathfrak{R}_2=\{b_1,b_2\}$
$\mathfrak{R}_{1'}=\{c_1,c_2,c_3\},\mathfrak{R}_{2'}=\{d_1,d_2\}$.
Let $(\mathfrak{Z},\mathfrak{L})$ and $(\mathfrak{Y},\mathfrak{M})$ be the classes of FHS sets. Let the $FHSs$ $(\tilde{I}_1,\Delta_1)$ over the universe $\mathfrak{Z}$ be defined as

$$(\tilde{I}_1,\Delta_1)=\begin{Bmatrix}\langle (a_1,b_1),\{\frac{\mathfrak{z}_1}{0.6},\frac{\mathfrak{z}_2}{0.8}\}\rangle,\\ \langle (a_2,b_1),\{\frac{\mathfrak{z}_1}{0.5},\frac{\mathfrak{z}_2}{0.7}\}\rangle,\\ \end{Bmatrix}$$

$\tau=\{\tilde{0}_{(\mathfrak{Z},\mathfrak{R})},\tilde{1}_{(\mathfrak{Z},\mathfrak{R})},(\tilde{I}_1,\Delta_1)\}$ is $FHSts$.
Let the $FHSs$'s $(\tilde{J}_1,\Delta_1),(\tilde{J}_2,\Delta_2),(\tilde{J}_3,\Delta_2),(\tilde{J}_4,\Delta_3),(\tilde{J}_5,\Delta_3),(\tilde{J}_6,\Delta_3)$ and $(\tilde{J}_7,\Delta_3)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1,\Delta_1)=\begin{Bmatrix}\langle (c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle (c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_2,\Delta_2)=\begin{Bmatrix}\langle (c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle (c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_3,\Delta_2)=\begin{Bmatrix}\langle (c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.7}\}\rangle,\\ \langle (c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.6}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_4,\Delta_3)=\begin{Bmatrix}\langle (c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle (c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle (c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_5,\Delta_3)=\begin{Bmatrix}\langle (c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle (c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle (c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_6, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.7}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.6}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_7, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.6}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.6}\} \rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{Y},\mathfrak{R})}, \tilde{1}_{(\mathfrak{Y},\mathfrak{R})}, (\tilde{J}_1, \Delta_1), (\tilde{J}_2, \Delta_2), (\tilde{J}_3, \Delta_2), (\tilde{J}_4, \Delta_3), (\tilde{J}_5, \Delta_3), (\tilde{J}_6, \Delta_3), (\tilde{J}_7, \Delta_3)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu): (\mathfrak{Z}, \mathfrak{L}) \to (\mathfrak{Y}, \mathfrak{M})$ be a FHS mapping as follows:

$\omega(\mathfrak{z}_1) = \mathfrak{y}_2, \omega(\mathfrak{z}_2) = \mathfrak{y}_1,$

$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2),\ \nu(a_1, b_2) = (c_2, d_2)$

$\mathfrak{h}(\tilde{I}_1, \Delta_1)^c = (\tilde{J}_1, \Delta_1)^c$

$(\tilde{I}_1, \Delta_1)^c$ is $FHScs$ in $\mathfrak{Z}$ and $\mathfrak{h}(\tilde{I}_1, \Delta_1)^c = (\tilde{J}_1, \Delta_1)^c$ is $FHScs$ in $\mathfrak{Y}$.

$\therefore$ $\mathfrak{h}$ is $FHSC$ mapping but not $FHS\theta C$ mapping because $(\tilde{I}_1, \Delta_1)$ is $FHScs$ in $\mathfrak{M}$ but $\mathfrak{h}(\tilde{I}_1, \Delta_1)^c = (\tilde{J}_1, \Delta_1)^c$ is not $FHS\theta cs$ in $\mathfrak{Y}$.

Example 3.4 *Let $\mathfrak{Z} = \{\mathfrak{z}_1, \mathfrak{z}_2\}$ and $\mathfrak{Y} = \{\mathfrak{y}_1, \mathfrak{y}_2\}$ be the FHS initial universes and the attributes be $\mathfrak{L} = \mathfrak{R}_1 \times \mathfrak{R}_2$ and $\mathfrak{M} = \mathfrak{R}_{1'} \times \mathfrak{R}_{2'}$ respectively. The attributes are given as:*

$\mathfrak{R}_1 = \{a_1, a_2, a_3\}, \mathfrak{R}_2 = \{b_1, b_2\}$

$\mathfrak{R}_{1'} = \{c_1, c_2, c_3\}, \mathfrak{R}_{2'} = \{d_1, d_2\}$.

Let $(\mathfrak{Z}, \mathfrak{L})$ and $(\mathfrak{Y}, \mathfrak{M})$ be the classes of FHS sets. Let the $FHSs$ $(\tilde{J}_1, \Delta_1)$ over the universe$\mathfrak{Y}$ be defined as

$$(\tilde{I}_1, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{z}_1}{0.6}, \frac{\mathfrak{z}_2}{0.8}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{z}_1}{0.5}, \frac{\mathfrak{z}_2}{0.7}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{z}_1}{0.4}, \frac{\mathfrak{z}_2}{0.5}\} \rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{Z},\mathfrak{L})}, \tilde{1}_{(\mathfrak{Z},\mathfrak{L})}, (\tilde{I}_1, \Delta_3)\}$ is $FHSts$.

Let the $FHSs$'s $(\tilde{J}_1, \Delta_1), (\tilde{J}_2, \Delta_2), (\tilde{J}_3, \Delta_2), (\tilde{J}_4, \Delta_3), (\tilde{J}_5, \Delta_3), (\tilde{J}_6, \Delta_3)$ and $(\tilde{J}_7, \Delta_3)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1, \Delta_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.6}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_2, \Delta_2) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.2}, \frac{\mathfrak{y}_2}{0.3}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_3, \Delta_2) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.7}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.6}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_4, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.6}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_5, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.2}, \frac{\mathfrak{y}_2}{0.3}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_6, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.7}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.6}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_7, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.6}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.6}\} \rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{Y},\mathfrak{R})}, \tilde{1}_{(\mathfrak{Y},\mathfrak{R})}, (\tilde{J}_1, \Delta_1), (\tilde{J}_2, \Delta_2), (\tilde{J}_3, \Delta_2), (\tilde{J}_4, \Delta_3), (\tilde{J}_5, \Delta_3), (\tilde{J}_6, \Delta_3), (\tilde{J}_7, \Delta_3)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu): (\mathfrak{Z}, \mathfrak{L}) \to (\mathfrak{Y}, \mathfrak{Z})$ be a FHS mapping as follows:

$\omega(\mathfrak{z}_1) = \mathfrak{y}_2,\ \omega(\mathfrak{z}_2) = \mathfrak{y}_1,$

$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2),\ \nu(a_1, b_2) = (c_2, d_2)$

$\mathfrak{h}(\tilde{I}_1, \Delta_3)^c = (\tilde{J}_4, \Delta_3)^c$

$(\tilde{I}_1, \Delta_3)^c$ is $FHScs$ in $\mathfrak{Z}$ and $\mathfrak{h}(\tilde{I}_1, \Delta_3)^c = (\tilde{J}_4, \Delta_3)^c$ is $FHSMcs$ in $\mathfrak{Y}$.

$\therefore$ $\mathfrak{h}$ is $FHSMC$ mapping but not $FHS\theta SC$ mapping because $(\tilde{I}_1, \Delta_1)^c$ is $FHScs$ in $\mathfrak{Z}$ but $\mathfrak{h}(\tilde{I}_1, \Delta_1)^c = (\tilde{J}_4, \Delta_1)^c$ is not $FHS\theta Scs$ in $\mathfrak{Y}$.

Example 3.5 *Let $\mathfrak{Z} = \{\mathfrak{z}_1, \mathfrak{z}_2\}$ and $\mathfrak{Y} = \{\mathfrak{y}_1, \mathfrak{y}_2\}$ be the FHS initial universes and the attributes be $\mathfrak{L} = \mathfrak{R}_1 \times \mathfrak{R}_2$ and $\mathfrak{M} = \mathfrak{R}_{1'} \times \mathfrak{R}_{2'}$ respectively. The attributes are given as:*

$\mathfrak{R}_1 = \{a_1, a_2, a_3\}, \mathfrak{R}_2 = \{b_1, b_2\}$

$\mathfrak{R}_{1'} = \{c_1, c_2, c_3\}, \mathfrak{R}_{2'} = \{d_1, d_2\}.$

Let $(\mathfrak{Z}, \mathfrak{L})$ and $(\mathfrak{Y}, \mathfrak{M})$ be the classes of FHS sets. Let the $FHSs$ $(\tilde{I}_1, \Delta_1)$ over the universe $\mathfrak{Z}$ be defined as

$$(\tilde{I}_1, \Delta_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{z}_1}{0.6}, \frac{\mathfrak{z}_2}{0.8}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{z}_1}{0.5}, \frac{\mathfrak{z}_2}{0.7}\} \rangle, \\ \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{Z},\mathfrak{R})}, \tilde{1}_{(\mathfrak{Z},\mathfrak{R})}, (\tilde{I}_1, \Delta_3)\}$ is $FHSts$.

Let the $FHSs$'s $(\tilde{J}_1, \Delta_1), (\tilde{J}_2, \Delta_2), (\tilde{J}_3, \Delta_2), (\tilde{J}_4, \Delta_3), (\tilde{J}_5, \Delta_3), (\tilde{J}_6, \Delta_3)$ and $(\tilde{J}_7, \Delta_3)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1, \Delta_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.6}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_2, \Delta_2) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.2}, \frac{\mathfrak{y}_2}{0.3}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_3, \Delta_2) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.7}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.6}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_4, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.6}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_5, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.2}, \frac{\mathfrak{y}_2}{0.3}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_6, \Delta_3) = \left\{\begin{matrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.7}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\}\rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.6}\}\rangle \end{matrix}\right\}$$

$$(\tilde{J}_7, \Delta_3) = \left\{\begin{matrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.6}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\}\rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.6}\}\rangle \end{matrix}\right\}$$

$\sigma = \{\tilde{0}_{(\mathfrak{Y},\mathfrak{R})}, \tilde{1}_{(\mathfrak{Y},\mathfrak{R})}, (\tilde{J}_1, \Delta_1), (\tilde{J}_2, \Delta_2), (\tilde{J}_3, \Delta_2), (\tilde{J}_4, \Delta_3), (\tilde{J}_5, \Delta_3), (\tilde{J}_6, \Delta_3), (\tilde{J}_7, \Delta_3)\}$ is $FHSts$.
Let $\mathfrak{h} = (\omega, \nu): (\mathfrak{Z}, \mathfrak{L}) \to (\mathfrak{Y}, \mathfrak{M})$ be a FHS mapping as follows:
$\omega(\mathfrak{z}_1) = \mathfrak{y}_2, \omega(\mathfrak{z}_2) = \mathfrak{y}_1$,
$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2),\ \nu(a_1, b_2) = (c_2, d_2)$
$\mathfrak{h}(\tilde{I}_1, \Delta_1)^c = (\tilde{J}_1, \Delta_1)^c$

$(\tilde{I}_1, \Delta_1)^c$ is $FHScs$ in $\mathfrak{Z}$ and $\mathfrak{h}(\tilde{I}_1, \Delta_1)^c = (\tilde{J}_1, \Delta_1)^c$ is $FHScs$ in $\mathfrak{Y}$.
$\therefore$ $\mathfrak{h}$ is $FHSC$ mapping but not $FHS\delta C$ mapping because $(\tilde{I}_1, \Delta_1)^c$ is $FHScs$ in $\mathfrak{Z}$ but $\mathfrak{h}(\tilde{I}_1, \Delta_1)^c = (\tilde{J}_1, \Delta_1)^c$ is not $FHS\delta cs$ in $\mathfrak{Y}$.

Example 3.6 *Let $\mathfrak{Z} = \{\mathfrak{z}_1, \mathfrak{z}_2\}$ and $\mathfrak{Y} = \{\mathfrak{y}_1, \mathfrak{y}_2\}$ be the FHS initial universes and the attributes be $\mathfrak{L} = \mathfrak{R}_1 \times \mathfrak{R}_2$ and $\mathfrak{M} = \mathfrak{R}_{1'} \times \mathfrak{R}_{2'}$, respectively. The attributes are given as:*
$\mathfrak{R}_1 = \{a_1, a_2, a_3\}, \mathfrak{R}_2 = \{b_1, b_2\}$
$\mathfrak{R}_{1'} = \{c_1, c_2, c_3\}, \mathfrak{R}_{2'} = \{d_1, d_2\}$.
Let $(\mathfrak{Z}, \mathfrak{L})$ and $(\mathfrak{Y}, \mathfrak{M})$ be the classes of FHS sets. Let the $FHSs$ $(\tilde{I}_1, \Delta_3)$ over the universe $\mathfrak{Z}$ be defined as

$$(\tilde{I}_1, \Delta_3) = \left\{\begin{matrix} \langle (a_1, b_1), \{\frac{\mathfrak{z}_1}{0.6}, \frac{\mathfrak{z}_2}{0.8}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{z}_1}{0.5}, \frac{\mathfrak{z}_2}{0.7}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{z}_1}{0.4}, \frac{\mathfrak{z}_2}{0.5}\}\rangle \end{matrix}\right\}$$

$\tau = \{\tilde{0}_{(\mathfrak{Z},\mathfrak{R})}, \tilde{1}_{(\mathfrak{Z},\mathfrak{R})}, (\tilde{I}_1, \Delta_3)\}$ is $FHSts$.
Let the $FHSs$'s $(\tilde{J}_1, \Delta_1), (\tilde{J}_2, \Delta_2), (\tilde{J}_3, \Delta_2), (\tilde{J}_4, \Delta_3), (\tilde{J}_5, \Delta_3), (\tilde{J}_6, \Delta_3)$ and $(\tilde{J}_7, \Delta_3)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1, \Delta_1) = \left\{\begin{matrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.6}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\}\rangle \end{matrix}\right\}$$

$$(\tilde{J}_2, \Delta_2) = \left\{\begin{matrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.2}, \frac{\mathfrak{y}_2}{0.3}\}\rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.4}\}\rangle \end{matrix}\right\}$$

$$(\tilde{J}_3, \Delta_2) = \left\{\begin{matrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.7}\}\rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.6}\}\rangle \end{matrix}\right\}$$

$$(\tilde{J}_4, \Delta_3) = \left\{\begin{matrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.6}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\}\rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.4}\}\rangle \end{matrix}\right\}$$

$$(\tilde{J}_5,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_6,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.7}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.6}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_7,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.6}\}\rangle\end{Bmatrix}$$

$\sigma=\{\tilde{0}_{(\mathfrak{Y},\mathfrak{R})},\tilde{1}_{(\mathfrak{Y},\mathfrak{R})},(\tilde{J}_1,\Delta_1),(\tilde{J}_2,\Delta_2),(\tilde{J}_3,\Delta_2),(\tilde{J}_4,\Delta_3),(\tilde{J}_5,\Delta_3),(\tilde{J}_6,\Delta_3),(\tilde{J}_7,\Delta_3)\}$ is $FHSts$.
Let $\mathfrak{h}=(\omega,\nu):(\mathfrak{Z},\mathfrak{L})\rightarrow(\mathfrak{Y},\mathfrak{M})$ be a FHS mapping as follows:
$\omega(\mathfrak{z}_1)=\mathfrak{y}_2,\omega(\mathfrak{z}_2)=\mathfrak{y}_1,$
$\nu(a_1,b_1)=(c_2,d_1),\nu(a_2,b_1)=(c_1,d_2),\ \nu(a_1,b_2)=(c_2,d_2)$
$\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_4,\Delta_3)^c$

$(\tilde{I}_1,\Delta_3)^c$ is $FHScs$ in $\mathfrak{Z}$ and $\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_4,\Delta_3)^c$ is $FHS\delta\mathcal{P}cs$ in $\mathfrak{Y}$.
$\therefore$ $\mathfrak{h}$ is $FHS\delta\mathcal{P}C$ mapping but not $FHS\delta C$ mapping because $(\tilde{I}_1,\Delta_3)^c$ is $FHScs$ in $\mathfrak{Z}$ but $\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_4,\Delta_3)^c$ is not $FHS\delta cs$ in $\mathfrak{Y}$.

Example 3.7 *Let $\mathfrak{Z}=\{\mathfrak{z}_1,\mathfrak{z}_2\}$ and $\mathfrak{Y}=\{\mathfrak{y}_1,\mathfrak{y}_2\}$ be the FHS initial universes and the attributes be $\mathfrak{L}=\mathfrak{R}_1\times\mathfrak{R}_2$ and $\mathfrak{M}=\mathfrak{R}_{1'}\times\mathfrak{R}_{2'}$, respectively. The attributes are given as:*
$\mathfrak{R}_1=\{a_1,a_2,a_3\},\mathfrak{R}_2=\{b_1,b_2\}$
$\mathfrak{R}_{1'}=\{c_1,c_2,c_3\},\mathfrak{R}_{2'}=\{d_1,d_2\}$.
Let $(\mathfrak{Z},\mathfrak{L})$ and $(\mathfrak{Y},\mathfrak{M})$ be the classes of FHS sets. Let the $FHSs$ $(\tilde{I}_1,\Delta_3)$ over the universe $\mathfrak{Z}$ be defined as

$$(\tilde{I}_1,\Delta_3)=\begin{Bmatrix}\langle(a_1,b_1),\{\frac{\mathfrak{z}_1}{0.3},\frac{\mathfrak{z}_2}{0.2}\}\rangle,\\ \langle(a_2,b_1),\{\frac{\mathfrak{z}_1}{0.5},\frac{\mathfrak{z}_2}{0.7}\}\rangle,\\ \langle(a_1,b_2),\{\frac{\mathfrak{z}_1}{0.4},\frac{\mathfrak{z}_2}{0.5}\}\rangle\end{Bmatrix}$$

$\tau=\{\tilde{0}_{(\mathfrak{Z},\mathfrak{R})},\tilde{1}_{(\mathfrak{Z},\mathfrak{R})},(\tilde{I}_1,\Delta_3)\}$ is $FHSts$.
Let the $FHSs$'s $(\tilde{J}_1,\Delta_1),(\tilde{J}_2,\Delta_2),(\tilde{J}_3,\Delta_2),(\tilde{J}_4,\Delta_3),(\tilde{J}_5,\Delta_3),(\tilde{J}_6,\Delta_3)$ and $(\tilde{J}_7,\Delta_3)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1,\Delta_1)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_2,\Delta_2)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_3,\Delta_2)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.7}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.6}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_4,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_5,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{m}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_6,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.7}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.6}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_7,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.6}\}\rangle\end{Bmatrix}$$

$\sigma=\{\tilde{0}_{(\mathfrak{Y},\mathfrak{R})},\tilde{1}_{(\mathfrak{Y},\mathfrak{R})},(\tilde{J}_1,\Delta_1),(\tilde{J}_2,\Delta_2),(\tilde{J}_3,\Delta_2),(\tilde{J}_4,\Delta_3),(\tilde{J}_5,\Delta_3),(\tilde{J}_6,\Delta_3),(\tilde{J}_7,\Delta_3)\}$ is $FHSts$.
Let $\mathfrak{h}=(\omega,\nu):(\mathfrak{Z},\mathfrak{L})\to(\mathfrak{Y},\mathfrak{M})$ be a FHS mapping as follows:
$\omega(\mathfrak{z}_1)=\mathfrak{y}_2,\omega(\mathfrak{z}_2)=\mathfrak{y}_1,$
$\nu(a_1,b_1)=(c_2,d_1),\nu(a_2,b_1)=(c_1,d_2),\ \nu(a_1,b_2)=(c_2,d_2)$
$\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_5,\Delta_3)^c$

$(\tilde{I}_1,\Delta_3)^c$ is $FHScs$ in $\mathfrak{Z}$ and $\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_5,\Delta_3)^c$ is $FHS\delta Scs$ in $\mathfrak{Y}$.
$\therefore$ $\mathfrak{h}$ is $FHS\delta SC$ mapping but not $FHS\delta C$ mapping because $(\tilde{I}_1,\Delta_3)^c$ is $FHScs$ in $\mathfrak{Z}$ but $\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_5,\Delta_3)^c$ is not $FHS\delta cs$ in $\mathfrak{Y}$.

Example 3.8 *Let $\mathfrak{Z}=\{\mathfrak{z}_1,\mathfrak{z}_2\}$ and $\mathfrak{Y}=\{\mathfrak{y}_1,\mathfrak{y}_2\}$ be the FHS initial universes and the attributes be $\mathfrak{L}=\mathfrak{R}_1\times\mathfrak{R}_2$ and $\mathfrak{M}=\mathfrak{R}_{1'}\times\mathfrak{R}_{2'}$ respectively. The attributes are given as:*
$\mathfrak{R}_1=\{a_1,a_2,a_3\},\mathfrak{R}_2=\{b_1,b_2\}$
$\mathfrak{R}_{1'}=\{c_1,c_2,c_3\},\mathfrak{R}_{2'}=\{d_1,d_2\}.$
Let $(\mathfrak{Z},\mathfrak{L})$ and $(\mathfrak{Y},\mathfrak{M})$ be the classes of FHS sets. Let the $FHSs$ $(\tilde{I}_1,\Delta_1)$ over the universe $\mathfrak{Z}$ be defined as

$$(\tilde{I}_1,\Delta_1)=\begin{Bmatrix}\langle(a_1,b_1),\{\frac{\mathfrak{z}_1}{0.4},\frac{\mathfrak{z}_2}{0.2}\}\rangle,\\ \langle(a_2,b_1),\{\frac{\mathfrak{z}_1}{0.5},\frac{\mathfrak{z}_2}{0.3}\}\rangle\end{Bmatrix}$$

$\tau=\{\tilde{0}_{(\mathfrak{Z},\mathfrak{R})},\tilde{1}_{(\mathfrak{Z},\mathfrak{R})},(\tilde{I}_1,\Delta_1)\}$ is $FHSts$.
Let the $FHSs$'s $(\tilde{J}_1,\Delta_1),(\tilde{J}_2,\Delta_2),(\tilde{J}_3,\Delta_2),(\tilde{J}_4,\Delta_3)$ and $(\tilde{J}_5,\Delta_1)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1,\Delta_1)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_2,\Delta_2)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_3,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_4,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.5}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_5,\Delta_1)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.4}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.3},\frac{\mathfrak{y}_2}{0.5}\}\rangle\end{Bmatrix}$$

$\sigma=\{\tilde{0}_{(\mathfrak{Y},\mathfrak{R})},\tilde{1}_{(\mathfrak{Y},\mathfrak{R})},(\tilde{J}_1,\Delta_1),(\tilde{J}_2,\Delta_2),(\tilde{J}_3,\Delta_2),(\tilde{J}_4,\Delta_3),(\tilde{J}_5,\Delta_1)\}$ is $FHSts$.
Let $\mathfrak{h}=(\omega,\nu):(\mathfrak{Z},\mathfrak{L})\rightarrow(\mathfrak{Y},\mathfrak{M})$ be a FHS mapping as follows:
$\omega(\mathfrak{z}_1)=\mathfrak{y}_2,\omega(\mathfrak{z}_2)=\mathfrak{y}_1,$
$\nu(a_1,b_1)=(c_2,d_1),\nu(a_2,b_1)=(c_1,d_2),\ \nu(a_1,b_2)=(c_2,d_2)$
$\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_5,\Delta_3)^c$

$(\tilde{I}_1,\Delta_3)^c$ is $FHScs$ in $\mathfrak{Z}$ and $\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_5,\Delta_3)^c$ is $FHSMcs$ in $\mathfrak{Y}$.
$\therefore$ $\mathfrak{h}$ is $FHSMC$ mapping but not $FHS\delta\mathcal{P}C$ mapping because $(\tilde{I}_1,\Delta_3)^c$ is $FHScs$ in $\mathfrak{Z}$ but $\mathfrak{h}(\tilde{I}_1,\Delta_3)^c=(\tilde{J}_5,\Delta_3)^c$ is not $FHS\delta\mathcal{P}cs$ in $\mathfrak{Y}$.

Example 3.9 *Let* $\mathfrak{Z}=\{\mathfrak{z}_1,\mathfrak{z}_2\}$ *and* $\mathfrak{Y}=\{\mathfrak{y}_1,\mathfrak{y}_2\}$ *be the* FHS *initial universes and the attributes be* $\mathfrak{L}=\mathfrak{R}_1\times\mathfrak{R}_2$ *and* $\mathfrak{M}=\mathfrak{R}_{1'}\times\mathfrak{R}_{2'}$ *respectively. The attributes are given as:*
$\mathfrak{R}_1=\{a_1,a_2,a_3\},\mathfrak{R}_2=\{b_1,b_2\}$
$\mathfrak{R}_{1'}=\{c_1,c_2,c_3\},\mathfrak{R}_{2'}=\{d_1,d_2\}.$
Let $(\mathfrak{Z},\mathfrak{L})$ and $(\mathfrak{Y},\mathfrak{M})$ be the classes of FHS sets. Let the $FHSs$ $(\tilde{I}_1,\Delta_1)$ over the universe $\mathfrak{Z}$ be defined as

$$(\tilde{I}_1,\Delta_1)=\begin{Bmatrix}\langle(a_1,b_1),\{\frac{\mathfrak{z}_1}{0.6},\frac{\mathfrak{z}_2}{0.8}\}\rangle,\\ \langle(a_2,b_1),\{\frac{\mathfrak{z}_1}{0.5},\frac{\mathfrak{z}_2}{0.7}\}\rangle\end{Bmatrix}$$

$\tau=\{\tilde{0}_{(\mathfrak{Z},\mathfrak{R})},\tilde{1}_{(\mathfrak{Z},\mathfrak{R})},(\tilde{I}_1,\Delta_3)\}$ is $FHSts$.
Let the $FHSs$'s $(\tilde{J}_1,\Delta_1),(\tilde{J}_2,\Delta_2),(\tilde{J}_3,\Delta_2),(\tilde{J}_4,\Delta_3),(\tilde{J}_5,\Delta_3),(\tilde{J}_6,\Delta_3)$ and $(\tilde{J}_7,\Delta_3)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1,\Delta_1)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_2,\Delta_2)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_3,\Delta_2)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.7}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.6}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_4,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_5,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{m}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_6,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.7}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{m}_1}{0.5},\frac{\mathfrak{m}_2}{0.6}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_7,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{m}_2}{0.6}\}\rangle\end{Bmatrix}$$

$\sigma=\{\tilde{0}_{(\mathfrak{N},Q)},\tilde{1}_{(\mathfrak{N},Q)},(\tilde{J}_1,\Delta_1),(\tilde{J}_2,\Delta_2),(\tilde{J}_3,\Delta_2),(\tilde{J}_4,\Delta_3),(\tilde{J}_5,\Delta_3),(\tilde{J}_6,\Delta_3),(\tilde{J}_7,\Delta_3)\}$ is $FHSts$.
Let $\mathfrak{h}=(\omega,\nu):(\mathfrak{Z},\mathfrak{L})\to(\mathfrak{Y},\mathfrak{M})$ be a FHS mapping as follows:
$\omega(\mathfrak{z}_1)=\mathfrak{y}_2,\omega(\mathfrak{z}_2)=\mathfrak{y}_1,$
$\nu(a_1,b_1)=(c_2,d_1),\nu(a_2,b_1)=(c_1,d_2),\ \nu(a_1,b_2)=(c_2,d_2)$
$\mathfrak{h}(\tilde{I}_1,\Delta_1)^c=(\tilde{J}_1,\Delta_1)^c$

$(\tilde{I}_1,\Delta_1)^c$ is $FHScs$ in $\mathfrak{Z}$ and $\mathfrak{h}(\tilde{I}_1,\Delta_1)^c=(\tilde{J}_1,\Delta_1)^c$ is $FHSecs$ in $\mathfrak{Y}$.
$\therefore$ $\mathfrak{h}$ is $FHSeC$ mapping but not $FHS\delta SC$ mapping because $(\tilde{I}_1,\Delta_1)^c$ is $FHSecs$ in $\mathfrak{Z}$ but $\mathfrak{h}(\tilde{I}_1,\Delta_1)^c=(\tilde{J}_1,\Delta_1)^c$ is not $FHS\delta Scs$ in $\mathfrak{Y}$.

Example 3.10 *Let $\mathfrak{Z}=\{\mathfrak{z}_1,\mathfrak{z}_2\}$ and $\mathfrak{Y}=\{\mathfrak{z}_1,\mathfrak{z}_2\}$ be the FHS initial universes and the attributes be $\mathfrak{L}=\mathfrak{R}_1\times\mathfrak{R}_2$ and $\mathfrak{M}=\mathfrak{R}_{1'}\times\mathfrak{R}_{2'}$, respectively. The attributes are given as:*
$\mathfrak{R}_1=\{a_1,a_2,a_3\},\mathfrak{R}_2=\{b_1,b_2\}$
$\mathfrak{R}_{1'}=\{c_1,c_2,c_3\},\mathfrak{R}_{2'}=\{d_1,d_2\}$.
Let $(\mathfrak{Z},\mathfrak{L})$ and $(\mathfrak{Y},\mathfrak{M})$ be the classes of FHS sets. Let the $FHSs$ $(\tilde{I}_1,\Delta_3)$ over the universe $\mathfrak{Z}$ be defined as

$$(\tilde{I}_1,\Delta_3)=\begin{Bmatrix}\langle(a_1,b_1),\{\frac{\mathfrak{y}_1}{0.4},\frac{\mathfrak{y}_2}{0.2}\}\rangle,\\ \langle(a_2,b_1),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle(a_1,b_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\end{Bmatrix}$$

$\tau=\{\tilde{0}_{(\mathfrak{Z},\mathfrak{R})},\tilde{1}_{(\mathfrak{Z},\mathfrak{R})},(\tilde{I}_1,\Delta_1)\}$ is $FHSts$.
Let the $FHSs$'s $(\tilde{J}_1,\Delta_1),(\tilde{J}_2,\Delta_2),(\tilde{J}_3,\Delta_2),(\tilde{J}_4,\Delta_3)$ and $(\tilde{J}_5,\Delta_3)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1,\Delta_1)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.8},\frac{\mathfrak{y}_2}{0.6}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_2,\Delta_2)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

$$(\tilde{J}_3,\Delta_3)=\begin{Bmatrix}\langle(c_2,d_1),\{\frac{\mathfrak{y}_1}{0.2},\frac{\mathfrak{y}_2}{0.3}\}\rangle,\\ \langle(c_1,d_2),\{\frac{\mathfrak{y}_1}{0.7},\frac{\mathfrak{y}_2}{0.5}\}\rangle,\\ \langle(c_2,d_2),\{\frac{\mathfrak{y}_1}{0.5},\frac{\mathfrak{y}_2}{0.4}\}\rangle\end{Bmatrix}$$

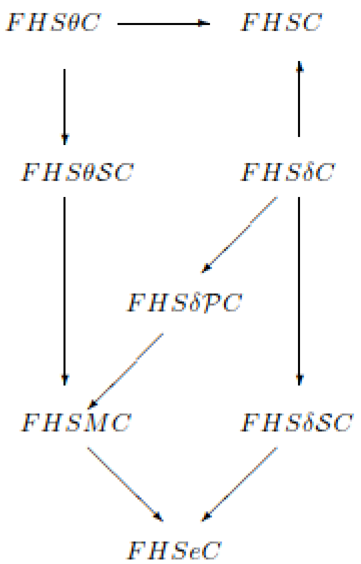
$$(\tilde{J}_4, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.8}, \frac{\mathfrak{y}_2}{0.6}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.7}, \frac{\mathfrak{y}_2}{0.5}\}\rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.5}\}\rangle \end{Bmatrix}$$

$$(\tilde{J}_5, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.2}, \frac{\mathfrak{y}_2}{0.4}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.3}, \frac{\mathfrak{y}_2}{0.5}\}\rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.5}\}\rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{Y},\mathfrak{R})}, \tilde{1}_{(\mathfrak{Y},\mathfrak{R})}, (\tilde{J}_1, \Delta_1), (\tilde{J}_2, \Delta_2), (\tilde{J}_3, \Delta_2), (\tilde{J}_4, \Delta_3)\}$ is $FHSts$.
Let $\mathfrak{h} = (\omega, \nu): (\mathfrak{Z}, \mathfrak{L}) \rightarrow (\mathfrak{Y}, \mathfrak{M})$ be a FHS mapping as follows:
$\omega(\mathfrak{z}_1) = \mathfrak{y}_2, \omega(\mathfrak{z}_2) = \mathfrak{y}_1,$
$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2), \ \nu(a_1, b_2) = (c_2, d_2)$
$\mathfrak{h}(\tilde{I}_1, \Delta_3)^c = (\tilde{J}_5, \Delta_3)^c$

$(\tilde{I}_1, \Delta_3)^c$ is $FHScs$ in $\mathfrak{Z}$ and $\mathfrak{h}(\tilde{I}_1, \Delta_3)^c = (\tilde{J}_5, \Delta_3)^c$ is $FHSecs$ in $\mathfrak{Y}$.
∴ $\mathfrak{h}$ is $FHSeC$ mapping but not $FHSMC$ mapping because $(\tilde{I}_1, \Delta_3)^c$ is $FHScs$ in $\mathfrak{Z}$ but $\mathfrak{h}(\tilde{I}_1, \Delta_3)^c = (\tilde{J}_5, \Delta_3)^c$ is not $FHSMcs$ in $\mathfrak{Y}$.

Remark 3.1 *From the results discussed above, the following diagram is obtained.*



Theorem 3.1 *A mapping* $\mathfrak{h}: (\mathfrak{Z}, \mathfrak{L}, \tau) \rightarrow (\mathfrak{Y}, \mathfrak{M}, \sigma)$ *is* $FHSMC$ *map iff for each* $FHSs(\tilde{J}, \Delta)$ *of* $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ *and for each* $FHSos$ $(\tilde{I}, \Delta)$ *of*$(\mathfrak{M}, L, \tau)$ *containing* $\mathfrak{h}^{-1}(\tilde{J}, \Delta)$, *there is a* $FHSMos$ $(\tilde{A}, \Delta)$ *of* $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ *such that* $(\tilde{J}, \Delta) \subseteq (\tilde{A}, \Delta)$ *and* $\mathfrak{h}^{-1}(\tilde{A}, \Delta) \subseteq (\tilde{I}, \Delta)$.

Proof. Necessity: Assume $\mathfrak{h}$ is a $FHSMC$ mapping. Let $(\tilde{J}, \Delta)$ be the $FHScs$ of $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ and $(\tilde{I}, \Delta)$ is a $FHSos$ of $(\mathfrak{Z}, \mathfrak{L}, \tau)$ such that $\mathfrak{h}^{-1}(\tilde{J}, \Delta) \subseteq (\tilde{I}, \Delta)$. Then, $(\tilde{A}, \Delta) = \tilde{1}_{(\mathfrak{Y},\mathfrak{R})} - \mathfrak{h}^{-1}((\tilde{I}, \Delta)^c)$ is $FHSMos$ of $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ such that $\mathfrak{h}^{-1}(\tilde{A}, \Delta) \subseteq (\tilde{I}, \Delta)$.
Sufficiency: Assume $(\tilde{I}, \Delta)$ is a $FHScs$ of $(\mathfrak{M}, L, \tau)$. Then, $(\mathfrak{h}(\tilde{I}, \Delta))^c$ is a $FHSs$ of $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ and $(\tilde{I}, \Delta)^c$ is $FHSos$ in $(\mathfrak{M}, L, \tau)$ such that $\mathfrak{h}^{-1}((\mathfrak{h}(\tilde{I}, \Delta))^c) \subseteq (\tilde{I}, \Delta)^c$. By hypothesis, there is a $FHSMos$

International Journal of Environmental Sciences
ISSN: 2229-7359
Vol. 11 No. 6s,2025
https://www.theaspd.com/ijes.php

$(\tilde{A},\Delta)$ of $(\mathfrak{Y},\mathfrak{M},\sigma)$ such that $(\mathfrak{h}(\tilde{I},\Delta))^c \subseteq (\tilde{A},\Delta)$ and $\mathfrak{h}^{-1}(\tilde{A},\Delta) \subseteq (\tilde{I},\Delta)^c$. Therefore, $(\tilde{I},\Delta) \subseteq (\mathfrak{h}^{-1}(\tilde{A},\Delta))^c$. Hence, $(\tilde{A},\Delta)^c \subseteq \mathfrak{h}(\tilde{I},\Delta) \subseteq \mathfrak{h}(\mathfrak{h}^{-1}(\tilde{A},\Delta))^c \subseteq (\tilde{A},\Delta)^c$ which implies $\mathfrak{h}(\tilde{I},\Delta) = (\tilde{A},\Delta)^c$. Since, $(\tilde{A},\Delta)^c$ is $FHSMcs$ of $(\mathfrak{Y},\mathfrak{M},\sigma)$, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHSMcs$ in $(\mathfrak{Y},\mathfrak{M},\sigma)$ and thus $\mathfrak{h}$ is $FHSMC$ mapping.

Theorem 3.2 *If $\mathfrak{h}: (\mathfrak{Z},\mathfrak{L},\tau) \to (\mathfrak{Y},\mathfrak{M},\sigma)$ is $FHSC$ and $\mathfrak{g}: (\mathfrak{Y},\mathfrak{M},\sigma) \to (P,N,\rho)$ is $FHSMC$ mappings, then $\mathfrak{g} \circ \mathfrak{h}: (\mathfrak{Z},\mathfrak{L},\tau) \to (P,N,\rho)$ is $FHSMC$ map.*

Proof. Let $(\tilde{I},\Delta)$ be a $FHScs$ in $(\mathfrak{Z},\mathfrak{L},\tau)$ then $\mathfrak{h}(\tilde{I},\Delta)$ is a $FHScs$ of $(\mathfrak{Y},\mathfrak{M},\sigma)$ because $\mathfrak{h}$ is a $FHSC$ mapping. Since $\mathfrak{g}$ is a $FHSMC$, $\mathfrak{g}(\mathfrak{h}(\tilde{I},\Delta)) = (\mathfrak{g} \circ \mathfrak{h})(\tilde{I},\Delta)$ is a $FHSMcs$ of (P,N,ρ). Hence, $\mathfrak{g} \circ \mathfrak{h}$ is $FHSMC$ mapping.

Theorem 3.3 *If $\mathfrak{h}: (\mathfrak{Z},\mathfrak{L},\tau) \to (\mathfrak{Y},\mathfrak{M},\sigma)$ is $FHSMC$ map, then $FHSMcl(\mathfrak{h}(\tilde{I},\Delta)) \subseteq \mathfrak{h}(FHScl(\tilde{I},\Delta))$.*

Proof. Let $\mathfrak{h}$ be a $FHSMC$ mapping and $(\tilde{I},\Delta)$ be a$FHScs$ in $(\mathfrak{Z},\mathfrak{L},\tau)$. Now $(\tilde{I},\Delta)) \subseteq FHScl(\tilde{I},\Delta)$ implies $\mathfrak{h}(\tilde{I},\Delta)) \subseteq \mathfrak{h}(FHScl(\tilde{I},\Delta))$. Since $\mathfrak{h}$ is a $FHSMC$ mapping, $\mathfrak{h}(FHScl(\tilde{I}))$ is $FHSMcs$ in $(\mathfrak{Y},\mathfrak{M},\sigma)$ such that $\mathfrak{h}: (\mathfrak{Z},\mathfrak{L},\tau) \subseteq \mathfrak{h}(FHScl(\tilde{I},\Delta))$. Therefore, $FHSMcl(\mathfrak{h}(\tilde{I},\Delta)) \subseteq \mathfrak{h}(FHScl(\tilde{I},\Delta))$.

Theorem 3.4 *Let $\mathfrak{h}: (\mathfrak{Z},\mathfrak{L},\tau) \to (\mathfrak{Y},\mathfrak{M},\sigma)$ and $\mathfrak{g}: (\mathfrak{Y},\mathfrak{M},\sigma) \to (P,N,\rho)$ be $FHSMC$ mappings. If every $FHSMcs$ of $(\mathfrak{Y},\mathfrak{M},\sigma)$ is $FHScs$, then $\mathfrak{g} \circ \mathfrak{h}: (\mathfrak{Z},\mathfrak{L},\tau) \to (P,N,\rho)$ is $FHSMC$.*

Proof. Let $(\tilde{I},\Delta)$ be a $FHScs$ in $(\mathfrak{Z},\mathfrak{L},\tau)$. Then, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHSMcs$ of $(\mathfrak{Y},\mathfrak{M},\sigma)$ because $\mathfrak{h}$ is $FHSMC$ mapping. By hypothesis, $\mathfrak{h}(\tilde{I},\Delta)$ is $FHScs$ of $(\mathfrak{Y},\mathfrak{M},\sigma)$. Now $\mathfrak{g}(\mathfrak{h}(\tilde{I},\Delta)) = (\mathfrak{g} \circ \mathfrak{h})(\tilde{I},\Delta)$ is $FHSMcs$ in (P,N,ρ) because $\mathfrak{g}$ is $FHSMC$ mapping. Thus, $\mathfrak{g} \circ \mathfrak{h}$ is $FHSMC$ mapping.

Theorem 3.5 *Let $\mathfrak{h}: (\mathfrak{Z},\mathfrak{L},\tau) \to (\mathfrak{Y},\mathfrak{M},\sigma)$ be a bijective mapping. Then the following statements are equivalent.*

1. $\mathfrak{h}$ is a $FHSMO$ mapping.
2. $\mathfrak{h}$ is a $FHSMC$ mapping.
3. $\mathfrak{h}^{-1}$ is a $FHSMCts$ mapping.

Proof. (1) ⇒ (2): Let us assume that $\mathfrak{h}$ is a $FHSMO$ mapping. By definition, if $(\tilde{I},\Delta)$ is a $FHSos$ in $(\mathfrak{Z},\mathfrak{L},\tau)$, then $\mathfrak{h}(\tilde{I},\Delta)$ is a $FHSMos$ in $(\mathfrak{Y},\mathfrak{M},\sigma)$. Hence, $(\tilde{I},\Delta)$ is $FHScs$ in $(\mathfrak{Z},\mathfrak{L},\tau)$. Then $\tilde{1}^c_{(\mathfrak{M},Q)} - (\tilde{I},\Delta)$ is a $FHSos$ in $(\mathfrak{Z},\mathfrak{L},\tau)$. By assumption, $\mathfrak{h}(\tilde{1}^c_{(\mathfrak{M},Q)} - (\tilde{I},\Delta))$ is a $FHSMos$ in $(\mathfrak{Y},\mathfrak{M},\sigma)$. Hence, $\tilde{1}^c_{(\mathfrak{N},Q)} - \mathfrak{h}(\tilde{1}^c_{(\mathfrak{M},Q)} - (\tilde{I},\Delta))$ is a $FHSMcs$ in $(\mathfrak{Y},\mathfrak{M},\sigma)$. Therefore, $\mathfrak{h}$ is a $FHSMC$ mapping.
(2) ⇒ (3): Let $(\tilde{I},\Delta)$ be a $FHScs$ in $(\mathfrak{Z},\mathfrak{L},\tau)$. By (ii), $\mathfrak{h}(\tilde{I},\Delta)$ is a $FHSMcs$ in $(\mathfrak{Y},\mathfrak{M},\sigma)$. Hence, $\mathfrak{h}(\tilde{I},\Delta) = (\mathfrak{h}^{-1})^{-1}(\tilde{I},\Delta)$. So $\mathfrak{h}^{-1}$ is a $FHSMcs$ in $(\mathfrak{Y},\mathfrak{M},\sigma)$. Hence, $\mathfrak{h}^{-1}$ is $FHSMCts$.
(3) ⇒ (1): Let $(\tilde{I},\Delta)$ be a $FHSos$ in $(\mathfrak{Z},\mathfrak{L},\tau)$. By (iii), $(\mathfrak{h}^{-1})^{-1}(\tilde{I},\Delta) = \mathfrak{h}(\tilde{I},\Delta)$ is a $FHSMO$ mapping.

4 Fuzzy hypersoft M- homeomorphism

In this section, we introduce the concept of fuzzy hypersoft M homeomorphism and discuss its properties.

Definition 4.1 *A bijection $\mathfrak{h}: (\mathfrak{Z},\mathfrak{L},\tau) \to (\mathfrak{Y},\mathfrak{M},\sigma)$ is called a fuzzy hypersoft homeomorphism (in short, $FHSHom$)(resp. fuzzy hypersoft M homeomorphism (in short, $FHSMHom$)) if $\mathfrak{h}$ and $\mathfrak{h}^{-1}$ are $FHSCts$ (resp. $FHSMCts$) mappings.*

Theorem 4.1 *Each $FHSHom$ is a $FHSMHom$. But not conversely.*

Proof. Let $\mathfrak{h}$ be $FHSHom$, then $\mathfrak{h}$ and $\mathfrak{h}^{-1}$ are $FHSCts$. But every $FHSCts$ function is

$FHSMCts$. Hence, $\mathfrak{h}$ and $\mathfrak{h}^{-1}$ are $FHSMCts$. Therefore, $\mathfrak{h}$ is a $FHSMHom$.

Example 4.1 *Let $\mathfrak{Z} = \{\mathfrak{z}_1, \mathfrak{z}_2\}$ and $\mathfrak{Y} = \{\mathfrak{y}_1, \mathfrak{y}_2\}$ be the FHS initial universes and the attributes be $\mathfrak{L} = \mathfrak{R}_1 \times \mathfrak{R}_2$ and $\mathfrak{M} = \mathfrak{R}_{1'} \times \mathfrak{R}_{2'}$, respectively. The attributes are given as:*

$\mathfrak{R}_1 = \{a_1, a_2, a_3\}, \mathfrak{R}_2 = \{b_1, b_2\}$

$\mathfrak{R}_{1'} = \{c_1, c_2, c_3\}, \mathfrak{R}_{2'} = \{d_1, d_2\}$.

Let $(\mathfrak{Z}, \mathfrak{L})$ and $(\mathfrak{Y}, \mathfrak{M})$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{I}_1, \Delta_1)$, $(\tilde{I}_2, \Delta_2)$, $(\tilde{I}_3, \Delta_2)$, $(\tilde{I}_4, \Delta_3)$ and $(\tilde{I}_5, \Delta_3)$ over the universe $\mathfrak{M}$ be

$$(\tilde{I}_1, \Delta_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{z}_1}{0.8}, \frac{\mathfrak{z}_2}{0.6}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{z}_1}{0.7}, \frac{\mathfrak{z}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{I}_2, \Delta_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{z}_1}{0.2}, \frac{\mathfrak{z}_2}{0.3}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{z}_1}{0.5}, \frac{\mathfrak{z}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{I}_3, \Delta_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{z}_1}{0.8}, \frac{\mathfrak{z}_2}{0.6}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{z}_1}{0.7}, \frac{\mathfrak{z}_2}{0.5}\} \rangle \\ \langle (a_1, b_2), \{\frac{\mathfrak{z}_1}{0.5}, \frac{\mathfrak{z}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{I}_4, \Delta_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{z}_1}{0.2}, \frac{\mathfrak{z}_2}{0.3}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{z}_1}{0.7}, \frac{\mathfrak{z}_2}{0.5}\} \rangle \\ \langle (a_1, b_2), \{\frac{\mathfrak{z}_1}{0.5}, \frac{\mathfrak{z}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{I}_5, \Delta_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{z}_1}{0.2}, \frac{\mathfrak{z}_2}{0.3}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{z}_1}{0.2}, \frac{\mathfrak{z}_2}{0.5}\} \rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{Z},\mathfrak{R})}, \tilde{1}_{(\mathfrak{Z},\mathfrak{R})}, (\tilde{I}_1, \Delta_1), (\tilde{I}_2, \Delta_2), (\tilde{I}_3, \Delta_3), (\tilde{I}_4, \Delta_3)\}$ is $FHSts$.

Let the $FHSs's$ $(\tilde{J}_1, \Delta_1), (\tilde{J}_2, \Delta_2), (\tilde{J}_3, \Delta_3), (\tilde{J}_4, \Delta_3)$ and $(\tilde{J}_5, \Delta_1)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1, \Delta_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.3}, \frac{\mathfrak{y}_2}{0.2}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.2}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_2, \Delta_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.6}, \frac{\mathfrak{y}_2}{0.8}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.7}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_3, \Delta_2) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.3}, \frac{\mathfrak{y}_2}{0.2}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_4, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.6}, \frac{\mathfrak{y}_2}{0.8}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{J}_5, \Delta_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{y}_1}{0.3}, \frac{\mathfrak{y}_2}{0.2}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.7}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{y}_1}{0.5}, \frac{\mathfrak{y}_2}{0.5}\} \rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{Y},\mathfrak{R})}, \tilde{1}_{(\mathfrak{Y},\mathfrak{R})}, (\tilde{J}_1, \Delta_1), (\tilde{J}_2, \Delta_2), (\tilde{J}_3, \Delta_2), (\tilde{J}_4, \Delta_3)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu): (\mathfrak{Z}, \mathfrak{L}) \to (\mathfrak{Y}, \mathfrak{M})$ be a FHS mapping as follows:

$\omega(\mathfrak{Z}_1) = \mathfrak{y}_2, \omega(\mathfrak{Z}_2) = \mathfrak{y}_1,$

$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2), \nu(a_1, b_2) = (c_2, d_2)$

$$\mathfrak{h}(\tilde{I}_1, \Delta_1) = (\tilde{J}_2, \Delta_1)$$
$$\mathfrak{h}(\tilde{I}_2, \Delta_2) = (\tilde{J}_3, \Delta_2)$$
$$\mathfrak{h}(\tilde{I}_3, \Delta_3) = (\tilde{J}_4, \Delta_3)$$
$$\mathfrak{h}(\tilde{I}_4, \Delta_3) = (\tilde{J}_5, \Delta_3)$$
$$\mathfrak{h}^{-1}(\tilde{J}_1, \Delta_1) = (\tilde{I}_5, \Delta_1)$$

Here $\mathfrak{h}^{-1}$ is $FHSMCts$ because $(\tilde{I}_1, \Delta_1)(\tilde{I}_2, \Delta_2), (\tilde{I}_3, \Delta_3)\&(\tilde{I}_4, \Delta_3)$ is $FHSos$ in $\mathfrak{Z}$ and $\mathfrak{h}(\tilde{I}_5, \Delta_1) = (\tilde{J}_1, \Delta_1)$ is $FHSMos$ in $\mathfrak{Y}$. Also, $\mathfrak{h}$ is $FHSMCts$ because $(\tilde{J}_1, \Delta_1)$ is $FHSos$ in $\mathfrak{Y}$ and $\mathfrak{h}^{-1}(\tilde{J}_1, \Delta_1)$ is $FHSMos$ in $\mathfrak{Z}$. Hence $\mathfrak{h}$ is $FHSMHom$. But $\mathfrak{h}$ is not $FHSHom$ because $(\tilde{J}_1, \Delta_1)$ is $FHSos$ in $\mathfrak{Y}$ but $\mathfrak{h}^{-1}(\tilde{J}_1, \Delta_1)$ is not $FHSos$ in $\mathfrak{Z}$.

Theorem 4.2 *Let $\mathfrak{h}: (\mathfrak{Z}, \mathfrak{L}, \tau) \to (\mathfrak{Y}, \mathfrak{M}, \sigma)$ be a bijective mapping. If $\mathfrak{h}$ is $FHSMCts$, then the following statements are equivalent:*

1. $\mathfrak{h}$ is a $FHSMC$ mapping.
2. $\mathfrak{h}$ is a $FHSMO$ mapping.
3. $\mathfrak{h}^{-1}$ is a $FHSMHom$.

Proof. (1)⇒(2): Assume that $\mathfrak{h}$ is a bijective mapping and a $FHSMC$ mapping. Here, $\mathfrak{h}^{-1}$ is a $FHSMCts$ mapping. We know that each $FHSos$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is a $FHSMos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$. Hence, $\mathfrak{h}$ is a $FHSMO$ mapping.
(2)⇒ (3): Let $\mathfrak{h}$ be a bijective and $FHSMO$ mapping. Further, $\mathfrak{h}$ is a $FHSMCts$ mapping. Hence, $\mathfrak{h}$ and $\mathfrak{h}^{-1}$ are $FHSMCts$. Therefore, $\mathfrak{h}$ is a $FHSMHom$.
(3)⇒(1): Let $\mathfrak{h}$ be a $FHSMHom$. Then $\mathfrak{h}$ and $\mathfrak{h}^{-1}$ are $FHSMCts$. Since each $FHScs$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$ is a $FHSMcs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$, $\mathfrak{h}$ is a $FHSMC$ mapping.

Theorem 4.3 *Let $\mathfrak{h}: (\mathfrak{Z}, \mathfrak{L}, \tau) \to (\mathfrak{Y}, \mathfrak{M}, \sigma)$ and $\mathfrak{g}: (\mathfrak{Y}, \mathfrak{M}, \sigma) \to (P, N, \rho)$ be two FHS mappings. Then the following hold:*

1. If $\mathfrak{g} \circ \mathfrak{h}$ is $FHSMO$ and $\mathfrak{h}$ is $FHSCts$, then $\mathfrak{g}$ is $FHSMO$.
2. If $\mathfrak{g} \circ \mathfrak{h}$ is $FHSO$ and $\mathfrak{g}$ is $FHSMCts$, then $\mathfrak{h}$ is $FHSMO$.

Proof.

1. Let $(\tilde{J}, \Delta)$ be a $FHSos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$. As $\mathfrak{h}$ is $FHSCts$ mapping, $\mathfrak{h}^{-1}(\tilde{J}, \Delta)$ is $FHSos$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$. As $\mathfrak{g} \circ \mathfrak{h}$ is $FHSMO$ mapping, $(\mathfrak{g} \circ \mathfrak{h})$ $(\mathfrak{h}^{-1}(\tilde{J}, \Delta))$ = $\mathfrak{g}(\mathfrak{h}(\mathfrak{h}^{-1}(\tilde{J}, \Delta)))$ = $\mathfrak{g}(\tilde{J}, \Delta)$ is $FHSMos$ in (P, N, ρ). Thus $\mathfrak{g}$ is $FHSMO$ mapping.

2. Let $(\tilde{J}, \Delta)$ be a $FHSos$ in (P, N, ρ). As $\mathfrak{g}$ is $FHSMCts$ mapping, $\mathfrak{g}^{-1}(\tilde{J}, \Delta)$ is $FHSMos$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$. As $\mathfrak{g} \circ \mathfrak{h}$ is $FHSO$ mapping, $(\mathfrak{g} \circ \mathfrak{h})(\mathfrak{g}^{-1}(\tilde{J}, \Delta))$ = $\mathfrak{h}(\mathfrak{g}(\mathfrak{g}^{-1}(\tilde{J}, \Delta)))$ = $\mathfrak{h}(\tilde{J}, \Delta)$ is $FHSMO$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$. This $\mathfrak{h}$ is $FHSMO$ mapping.

5 Fuzzy hypersoft *M- C* homeomorphism

In this section we introduce fuzzy hypersoft *M-C* homeomorphism and analyse some of its properties.

Definition 5.1 *A bijection $\mathfrak{h}: (\mathfrak{Z}, \mathfrak{L}, \tau) \to (\mathfrak{Y}, \mathfrak{M}, \sigma)$ is called a fuzzy hypersoft M- C homeomorphism (in short, $FHSMCHom$) if $\mathfrak{h}$ and $\mathfrak{h}^{-1}$ are $FHSMIrr$ mappings.*

Theorem 5.1 *Each $FHSMCHom$ is a $FHSMHom$. But the converse is not true.*

Proof. Let us assume that $(\tilde{J}, \Delta)$ is a $FHScs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$. This shows that $(\tilde{J}, \Delta)$ is a $FHSMcs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$. By assumption, $\mathfrak{h}^{-1}(\tilde{J}, \Delta)$ is a $FHSMcs$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$. Hence, $\mathfrak{h}$ is a $FHSMCts$ mapping. Therefore, $\mathfrak{h}$ and $\mathfrak{h}^{-1}$ are $FHSMCts$ mappings. Hence, $\mathfrak{h}$ is a $FHSMHom$.

Example 5.1 *Let $\mathfrak{Z} = \{\mathfrak{z}_1, \mathfrak{z}_2\}$ and $\mathfrak{Y} = \{\mathfrak{y}_1, \mathfrak{y}_2\}$ be the FHS initial universes and the attributes be $\mathfrak{L} =$*

$\mathfrak{R}_1 \times \mathfrak{R}_2$ *and* $\mathfrak{M} = \mathfrak{R}_{1'} \times \mathfrak{R}_{2'}$ *respectively. The attributes are given as:*
$\mathfrak{R}_1 = \{a_1, a_2\}, \mathfrak{R}_2 = \{b_1\}$
$\mathfrak{R}_{1'} = \{c_1, c_2\}, \mathfrak{R}_{2'} = \{d_1\}.$
Let $(\mathfrak{Z}, \mathfrak{L})$ and $(\mathfrak{Y}, \mathfrak{N})$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{I}_1, \Delta)$, $(\tilde{I}_2, \Delta)$ and $(\tilde{I}_3, \Delta)$ over the universe $\mathfrak{M}$ be

$$(\tilde{I}_1, \Delta) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.1}, \frac{\mathfrak{m}_2}{0.2}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.3}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$
$$(\tilde{I}_2, \Delta) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.9}, \frac{\mathfrak{m}_2}{0.8}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$
$$(\tilde{I}_3, \Delta) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.5}\}\rangle \\ \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{Z},\mathfrak{N})}, \tilde{1}_{(\mathfrak{Y},\mathfrak{N})}, (\tilde{I}_1, \Delta), (\tilde{I}_2, \Delta)\}$ is $FHSts$.
Let the $FHSs's$ $(\tilde{J}_1, \Delta), (\tilde{J}_2, \Delta)$, and $(\tilde{J}_3, \Delta)$ over the universe $\mathfrak{Y}$ be

$$(\tilde{J}_1, \Delta) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.2}, \frac{\mathfrak{n}_2}{0.1}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.4}, \frac{\mathfrak{n}_2}{0.5}\}\rangle \end{Bmatrix}$$
$$(\tilde{J}_2, \Delta) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.8}, \frac{\mathfrak{n}_2}{0.9}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.6}, \frac{\mathfrak{n}_2}{0.7}\}\rangle \end{Bmatrix}$$
$$(\tilde{J}_3, \Delta) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.5}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.5}\}\rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{Y},\mathfrak{N})}, \tilde{1}_{(\mathfrak{Y},\mathfrak{N})}, (\tilde{J}_1, \Delta), (\tilde{J}_2, \Delta), (\tilde{J}_3, \Delta)\}$ is $FHSts$.
Let $\mathfrak{h} = (\omega, \nu): (\mathfrak{Z}, \mathfrak{N}) \to (\mathfrak{Y}, \mathfrak{N})$ be a FHS mapping as follows:
$\omega(\mathfrak{z}_1) = \mathfrak{y}_2, \omega(\mathfrak{z}_2) = \mathfrak{y}_1,$

$$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2)$$
$$\mathfrak{h}(\tilde{I}_1, \Delta) = (\tilde{J}_1, \Delta)$$
$$\mathfrak{h}(\tilde{I}_2, \Delta) = (\tilde{J}_2, \Delta)$$
$$\mathfrak{h}(\tilde{I}_3, \Delta) = (\tilde{J}_3, \Delta)$$

Here $\mathfrak{h}$ is $FHSMHom$ but not $FHSMCHom$ because $(\tilde{J}_3, \Delta)$ is $FHSMos$ in $\mathfrak{N}$ but $\mathfrak{h}^{-1}(\tilde{J}_3, \Delta) = (\tilde{I}_3, \Delta)$is not $FHSMos$ in $\mathfrak{Z}$.

Theorem 5.2 *If* $\mathfrak{h}: (\mathfrak{Z}, \mathfrak{L}, \tau) \to (\mathfrak{Y}, \mathfrak{M}, \sigma)$ *is a* $FHSMCHom$, *then* $FHSMcl(\mathfrak{h}^{-1}(\tilde{J}, \Delta)) \subseteq \mathfrak{h}^{-1}(FHScl(\tilde{J}, \Delta))$ *for each* $FHSs$ $(\tilde{J}, \Delta)$ *in* $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.

Proof. Let $(\tilde{J}, \Delta)$ be a $FHSs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$. Then, $FHScl(\tilde{J}, \Delta)$ is a $FHScs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$ and every $FHScs$ is a $FHSMcs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$. Assume $\mathfrak{h}$ is $FHSMIrr$ and $\mathfrak{h}^{-1}(FHScl(\tilde{J}, \Delta))$ is a $FHSMcs$ in $(\mathfrak{Z}, \mathfrak{L}, \tau)$. Then $FHSMcl(\mathfrak{h}^{-1}(FHScl(\tilde{J}, \Delta))) = \mathfrak{h}^{-1}(FHScl(\tilde{J}, \Delta))$. Here, $FHSMcl(\mathfrak{h}^{-1}(\tilde{J}, \Delta)) \subseteq FHSMcl(\mathfrak{h}^{-1}(FHScl(\tilde{J}, \Delta))) = \mathfrak{h}^{-1}(FHScl(\tilde{J}, \Delta))$. Therefore,
$FHSMcl(\mathfrak{h}^{-1}(\tilde{J}, \Delta)) \subseteq \mathfrak{h}^{-1}(FHScl(\tilde{J}, \Delta))$ for every $FHSs$ $(\tilde{J}, \Delta)$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.

Theorem 5.3 *Let* $\mathfrak{h}: (\mathfrak{Z}, \mathfrak{L}, \tau) \to (\mathfrak{Y}, \mathfrak{M}, \sigma)$ *be a* $FHSMCHom$. *Then* $FHSMcl(\mathfrak{h}^{-1}(\tilde{J}, \Delta)) = \mathfrak{h}^{-1}(FHSMcl(\tilde{J}, \Delta))$ *for each* $FHSs$ $(\tilde{J}, \Delta)$ *in* $(\mathfrak{Y}, \mathfrak{M}, \sigma)$.

Proof. Since $\mathfrak{h}$ is a $FHSMCHom$, $\mathfrak{h}$ is a $FHSMIrr$ mapping. Let $(\tilde{J}, \Delta)$ be a $FHSs$ in $(\mathfrak{N}, M, \sigma)$. Clearly, $FHSMcl(\tilde{J}, \Delta)$ is a $FHSMcs$ in $(\mathfrak{Y}, \mathfrak{M}, \sigma)$. Since, $\mathfrak{h}^{-1}(\tilde{J}, \Delta) \subseteq \mathfrak{h}^{-1}(FHSMcl(\tilde{J}, \Delta))$, $FHSMcl(\mathfrak{h}^{-1}(\tilde{J}, \Delta)) \subseteq FHSMcl(\mathfrak{h}^{-1}(FHSMcl\ (\tilde{J}, \Delta))) = \mathfrak{h}^{-1}(FHSMcl(\tilde{J}, \Delta))$. Therefore,

$FHSMcl(\mathfrak{h}^{-1}(\tilde{J},\Delta)) \subseteq \mathfrak{h}^{-1}(FHSMcl(\tilde{J},\Delta))$. Let $\mathfrak{h}$ be a $FHSMCHom$. $\mathfrak{h}^{-1}$ is a $FHSMIrr$ mapping. Let us consider $FHSs$ $\mathfrak{h}^{-1}(\tilde{J},\Delta)$ in $(\mathfrak{Z},\mathfrak{L},\tau)$, which implies $FHSMcl(\mathfrak{h}^{-1}(\tilde{J},\Delta))$ is a $FHSMcs$ in $(\mathfrak{Z},\mathfrak{L},\tau)$. Hence, $FHSMcl(\mathfrak{h}^{-1}(\tilde{J},\Delta))$ is a $FHSMcs$ in $(\mathfrak{Z},\mathfrak{L},\tau)$. This implies that $(\mathfrak{h}^{-1})^{-1}(FHSMcl(\mathfrak{h}^{-1}(\tilde{J},\Delta))) = \mathfrak{h}(FHSMcl(\mathfrak{h}^{-1}(\tilde{J},\Delta)))$ is a $FHSMcs$ in $(\mathfrak{Y},\mathfrak{M},\sigma)$. This proves $(\tilde{J},\Delta) = (\mathfrak{h}^{-1})^{-1}(\mathfrak{h}^{-1}(\tilde{J},\Delta)) \subseteq (\mathfrak{h}^{-1})^{-1}(FHSMcl(\mathfrak{h}^{-1}\ (\tilde{J},\Delta))) = \mathfrak{h}(FHSMcl(\mathfrak{h}^{-1}(\tilde{J},\Delta)))$. Therefore, $FHSMcl(\tilde{J},\Delta) \subseteq FHSMcl(\mathfrak{h}(FHSMcl(\mathfrak{h}^{-1}(\tilde{J},\Delta)))) = \mathfrak{h}(FHSMcl(\mathfrak{h}^{-1}\ (\tilde{J},\Delta)))$, since $\mathfrak{h}^{-1}$ is a $FHSMIrr$ mapping. Hence, $\mathfrak{h}^{-1}(FHSMcl(\tilde{J},\Delta)) \subseteq \mathfrak{h}^{-1}(\mathfrak{h}(FHSMcl(\mathfrak{h}^{-1}(\tilde{J},\Delta))) = FHSMcl(\mathfrak{h}^{-1}(\tilde{J},\Delta))$. That is, $\mathfrak{h}^{-1}(FHSMcl(\tilde{J},\Delta)) \subseteq FHSMcl(\mathfrak{h}^{-1}(\tilde{J},\Delta))$. Hence, $FHSMcl(\mathfrak{h}^{-1}(\tilde{J},\Delta)) = \mathfrak{h}^{-1}(FHSMcl(\tilde{J},\Delta))$.

Theorem 5.4 *If* $\mathfrak{h}: (\mathfrak{Z},\mathfrak{L},\tau) \to (\mathfrak{Y},\mathfrak{M},\sigma)$ *and* $\mathfrak{g}: (\mathfrak{Y},\mathfrak{M},\sigma) \to (P,N,\rho)$ *are* $FHSMCHom$*'s, then* $\mathfrak{g} \circ \mathfrak{h}$ *is a* $FHSMCHom$.

Proof. Let $\mathfrak{h}$ and $\mathfrak{g}$ be two $FHSMCHom$'s. Assume $(\tilde{J},\Delta)$ is a $FHSMcs$ in (P,N,ρ). Then, $\mathfrak{g}^{-1}(\tilde{J},\Delta)$ is a $FHSMcs$ in $(\mathfrak{Y},\mathfrak{M},\sigma)$. Then, by hypothesis, $\mathfrak{h}^{-1}(\mathfrak{g}^{-1}(\tilde{J},\Delta))$ is a $FHSMcs$ in $(\mathfrak{Z},\mathfrak{L},\tau)$. Hence, $\mathfrak{g} \circ \mathfrak{h}$ is a $FHSMIrr$ mapping. Now, let $(\tilde{I},\Delta)$ be a $FHSMcs$ in $(\mathfrak{Z},\mathfrak{L},\tau)$. Then, by presumption, $\mathfrak{h}(\mathfrak{g})$ is a $FHSMcs$ in $(\mathfrak{Y},\mathfrak{M},\sigma)$. Then, by hypothesis, $\mathfrak{g}(\mathfrak{h}(\tilde{I},\Delta))$ is a $FHSMcs$ in (P,N,ρ). This implies that $\mathfrak{g} \circ \mathfrak{h}$ is a $FHSMIrr$ mapping. Hence, $\mathfrak{g} \circ \mathfrak{h}$ is a $FHSMCHom$.

CONCLUSION

In this paper, we introduce the notion of fuzzy hypersoft M-closed maps ($FHSMC$ maps) and investigate their fundamental properties, supported by illustrative examples. The study then proceeds to compare $FHSMC$ maps with several existing classes of maps, including $FHS\theta C$, $FHS\theta SC$, $FHS\delta O$, $FHS\delta\mathcal{P}C$, $FHS\delta SC$ and $FHSeC$ maps. These comparisons highlight the relationships, distinctions, and relative strengths of the $FHSMC$ maps within the broader landscape of fuzzy hypersoft topological mappings.

Furthermore, the concept is extended to include fuzzy hypersoft homeomorphisms and fuzzy hypersoft M-homeomorphisms, providing a deeper understanding of topological equivalences in fuzzy hypersoft settings. In addition, the study introduces the notion of fuzzy hypersoft M-C homeomorphisms and explores several of their key properties and characterizations.

This foundational work opens several avenues for future research. Specifically, these findings may be extended to explore the concepts of fuzzy hypersoft contra-MM-open mappings, fuzzy hypersoft contra M-closed mappings, fuzzy hypersoft contra M-homeomorphisms, and fuzzy hypersoft contra M-C homeomorphisms, which could provide further insights into the dual behavior of mappings in fuzzy hypersoft topological spaces.

References

[1] M. Abbas, G. Murtaza and F. Smarandache, *Basic operations on hypersoft sets and hypersoft point*, Neutrosophic Sets and Systems, **35**, (2020), 407-421.
[2] M. Ahsan, M. Saeed and A.U. Rahman, *A Theoretical and Analytical Approach for Fundamental Framework of Composite Mappings on Fuzzy Hypersoft Classes*, Neutrosophic Sets and Systems, **45**, (2021), 268-285.
[3] D. Ajay and J. Joseline Charisma, *Neutrosophic hypersoft topological spaces*, Neutrosophic Sets and Systems, **40**, (2021), 178-194.
[4] D. Ajay, J. Joseline Charisma, N. Boonsatit, P. Hammachukiattikul and G. Rajchakit *Neutrosophic semiopen hypersoft sets with an application to MAGDM under the COVID-19 scenario*, Hindawi Journal of Mathematics, **2021**, (2021), 1-16.
[5] C. G. Aras and S. Bayramov, *Neutrosophic Soft Continuity in Neutrosophic Soft Topological Spaces*, Filomat, **34:10**, (2020), 3495 - 3506.
[6] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182-190.
[7] D. Molodtsov, *Soft set theory-first results*, Comput. Math. Appl., **37**, (1999), 19-31.
[8] M. Shabir and M. Naz, *On soft topological spaces*, Comput. Math. Appl., **61**, (2011), 1786-1799.
[9] F. Smarandache, *Extension of soft set to hypersoft set, and then to plithogenic hypersoft set* , Neutrosophic Sets and Systems, **22**, (2018), 168-170.
[10] P. Surendra, K. Chitirakala and A. Vadivel, *δ-open sets in neutrosophic hypersoft topological spaces*, International Journal of Neutrosophic Science, **20** (4), (2023), 93-105.
[11] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (3), (1965), 338â€“353.